

# INVARIANTS OF MODELS OF GENUS ONE CURVES VIA MODULAR FORMS AND DETERMINANTAL REPRESENTATIONS

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**ABSTRACT.** An invariant of a model of genus one curve is a polynomial in the coefficients of the model that is stable under certain linear transformations. The classical example of an invariant is the discriminant, which characterizes the singularity of models. The ring of invariants of genus one models over a field is generated by two elements. Fisher normalized these invariants for models of degree  $n = 2, 3, 4$  in such a way that these invariants are moreover defined over the integers. We will provide an alternative way to express these normalized invariants using modular forms. This method relies on a direct computation for the discriminants based on their own geometric properties. In the case of the discriminant of ternary cubics over the complex numbers, we perform another approach using determinantal representations with a connection to theta functions. Both of these two approaches link a genus one model to a Weierstrass form.

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## 1. INTRODUCTION

Consider a curve  $C$  given by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \quad (1)$$

There are two classical invariants labeled by  $c_4$  and  $c_6$  defined for instance in [19, p. 42] as

$$c_4 = b_2^2 - 24b_4, c_6 = -b_2^3 + 36b_2b_4 - 216b_6, \quad (2)$$

where  $b_2 = a_1^2 + 4a_2$ ,  $b_4 = 2a_4 + a_1a_3$  and  $b_6 = a_3^2 + 4a_6$ . They define the discriminant  $\Delta = (c_4^3 - c_6^2)/1728$  with the property that  $\Delta \neq 0$  if and only if the curve is non-singular. In this case, the curve  $C$  is of genus one. An invariant of the model associated to (1) is a polynomial in their coefficients, which remains unchanged under the linear algebraic transformations mentioned in Section 2.

We want to explore this study to other models of genus one curves with two different approaches. On the one hand, it is natural to relate invariants to modular forms as described in Section 4. On the other hand, one can represent a polynomial of certain types as the determinant of a matrix whose elements have linear forms. The study of a curve defined by that polynomial is reduced to the study of the corresponding matrix.

**1.1. Models of genus one and their invariants.** Let  $C$  be a smooth curve of genus one over a field  $K$  and suppose that  $D$  is a  $K$ -rational divisor on  $C$  of degree  $n$ . In case  $n = 1$ ,  $C$  has a  $K$ -rational point so that it can be given by a Weierstrass equation (1).

If  $n \geq 2$ , there exists a morphism  $C \rightarrow \mathbb{P}^{n-1}$  defined by the complete linear system associated to  $D$ . This morphism is an embedding if  $n \geq 3$ . We define models of genus one of degrees  $n \leq 5$  as follows

**Definition.** A genus one model of degree  $n \leq 5$  is a

- a) Weierstrass form if  $n = 1$ .
- b) pair of a binary quadratic and a binary quartic if  $n = 2$ .
- c) ternary cubic if  $n = 3$ .
- d) pair of quadrics in four variables if  $n = 4$ .
- e)  $5 \times 5$  alternating matrix of linear forms in five variables if  $n = 5$ .

The equation defined by a genus one model  $(p, q)$  of degree  $n = 2$  is  $y^2 + p(x, z)y = q(x, z)$ . In case  $n = 5$ , the equations defining the model are the  $4 \times 4$  Pfaffians of the matrix. In general, such models define smooth curves of genus one.

The classical invariants where  $n = 2, 3, 4$  were studied in [23] and [1, Section 3]. These invariants were normalized by Fisher [12, Sections 6,7] so that they are usual formulae when restricted to the Weierstrass family.

The aim of this paper is to give an alternative way to express the normalized invariants  $c_4$ ,  $c_6$  and thus  $\Delta = (c_4^3 - c_6^2)/1728$  for genus one models of degrees  $n = 2, 3, 4$ . To do this, we establish formulae in all characteristics relating the invariants of smooth genus one models of degrees  $n \leq 4$  and the corresponding Jacobians in the classical setting as in [1]. More precisely, the authors in [1, Section 3] define a map  $f_n$  from a smooth curve  $C_\phi$ , which is defined by a model of genus one  $\phi$  of degree  $n$  ( $n = 2, 3, 4$ ), to the corresponding Jacobian  $E_\phi$ . In addition, they describe explicitly when  $\text{char}(K) \neq 2, 3$  the map and the Jacobian  $E_\phi$  given by a Weierstrass equation of the form (3).

The map  $f_n$  will be described explicitly in Section 2. We construct from it in Section 3 the map  $\varphi_n : X_n \rightarrow W$  from the affine space  $X_n$  of genus one models of degree  $n$  ( $n = 2, 3, 4$ ) to the space  $W$  of Weierstrass forms. We first compute the normalized discriminants.

**Theorem 1.1.** *Let  $C_\phi$  be a curve defined by a genus one model  $\phi$  of degree  $n$  ( $n = 2, 3, 4$ ) over a field  $K$  and  $\Delta_\phi, \Delta_{\varphi_n(\phi)}$  be the discriminants of  $\phi$  and its corresponding Weierstrass form  $\varphi_n(\phi)$ . We have*

$$\Delta_\phi = \alpha_n^{12} \Delta_{\varphi_n(\phi)},$$

where  $\alpha_2 = 1, \alpha_3 = 1/2, \alpha_4 = 2$ .

This theorem is established directly using the singularities of genus one models. To obtain the analogous result for any invariant, we will use geometric modular forms defined in [18]. To be precise, we will see in Section 4 that it is possible to associate to a geometric modular form  $\mathcal{F}$  an invariant  $I_{\mathcal{F}}$  of the same weight. We will prove the following result

**Theorem 1.2.** *Let  $C_\phi$  be a smooth curve of genus one defined by a model  $\phi$  of degree  $n$  ( $n = 2, 3, 4$ ) over a field  $K$  with the corresponding Jacobian  $E_\phi$  defined by  $\varphi_n(\phi)$ . Let  $k$  be an integer and  $I_{\mathcal{F}}$  be the invariant of weight  $k$  associated to a geometric modular form  $\mathcal{F}$  of weight  $k$ , we have*

$$I_{\mathcal{F}}(\phi) = \alpha_n^k I_{\mathcal{F}}(\varphi_n(\phi)),$$

where  $\alpha_2 = 1, \alpha_3 = 1/2, \alpha_4 = 2$ .

Recently, Fisher [13, p. 2126] have obtained a formula for the invariants  $c_4, c_6$  and the Jacobian of smooth genus one models of arbitrary degree  $n$  in characteristic 0. More details about models of genus one curves and their invariants will be discussed in Section 2.

**1.2. Invariants over  $\mathbb{C}$  and determinantal representations.** The second perspective of this paper is to study the invariants of genus one models when  $K = \mathbb{C}$  with an emphasis on discriminants of ternary cubics. These invariants have large expressions in general. For instance, the discriminant of a plane cubic curve is a polynomial of degree 12 in coefficients of the cubic with 2040 monomials (see [15, p. 4]). But over  $\mathbb{C}$ , we have short expressions in terms of theta constants. Consider the classical case where our smooth projective genus one curve  $C_\phi$  is defined by the affine Weierstrass equation:

$$y^2 = 4x^3 - g_2x - g_3. \tag{3}$$

Using the Weierstrass parametrization, there exists a unique lattice  $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$  with some complex numbers  $\omega_1, \omega_2$  such that  $\text{Im}(\omega_2/\omega_1) > 0$  and  $C_\phi(\mathbb{C}) \cong \mathbb{C}/\Lambda$ . Let  $\tau := \omega_2/\omega_1$  and apply the discriminant formula  $\Delta_\phi = 2^{12}(g_2^3 - 27g_3^2)$ , we have that

$$\Delta_\phi = 2^{16} \left( \frac{\pi}{\omega_1} \right)^{12} (\theta_2(0, \tau)\theta_3(0, \tau)\theta_4(0, \tau))^8. \quad (4)$$

Here  $\theta_2, \theta_3$  and  $\theta_4$  are the three even Jacobi theta functions.

**Remark 1.3.** *The normalized discriminant above comes from the formulae in [2, p. 367-368] with the normalized invariants  $c_4 = 2^6 3g_2$ ,  $c_6 = 2^9 3^3 g_3$ .*

More details on the above will be explained in Section 5. Our purpose is to generalize the formula (4) to other models of genus one. But this is an immediate consequence of Theorem 1.1. We have the following formulae for the discriminants of genus one models of degree  $n = 2, 3, 4$ . Again, a lattice  $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$  associated to a Weierstrass form over  $\mathbb{C}$  will be understood as the unique one coming from the Weierstrass parametrization.

**Corollary 1.4.** *Let  $C_\phi$  be a smooth curve of genus one over  $\mathbb{C}$  defined by a model  $\phi$  of degree  $n$  ( $n = 2, 3, 4$ ) with the Jacobian  $E_\phi$  defined by  $\varphi_n(\phi)$ . Then  $E_\phi(\mathbb{C}) \cong \mathbb{C}/\Lambda$  for the lattice  $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$  with some complex numbers  $\omega_1, \omega_2$  satisfying  $\text{Im}(\omega_2/\omega_1) > 0$ . Let  $\Delta_\phi$  be the discriminant of  $\phi$ , we then have*

$$\Delta_\phi = 2^{16} \alpha_n^{12} \left( \frac{\pi}{\omega_1} \right)^{12} (\theta_2(0, \tau)\theta_3(0, \tau)\theta_4(0, \tau))^8,$$

where  $\alpha_2 = 1, \alpha_3 = 1/2$  and  $\alpha_4 = 2$ .

We want to study the above discriminant formula with a new approach using determinantal representations. For a homogeneous polynomial  $\phi$ , we construct a matrix  $U$  whose elements are linear forms such that we can write  $\phi = \lambda \det(U)$  for some constant  $\lambda \neq 0$ . In general, only plane curves, quadratic and cubic surfaces, quadratic three-folds admit a determinantal representation as confirmed in [10]. The study of  $\phi$  has thus been moved to the study of the matrix  $U$ . The reader can have a look at [4] for a general discussion of this topic.

Starting with Weierstrass cubics, we find theta functions in their determinantal representations as well as in the discriminants. Let

$$a = \theta_2(0, \tau), \quad b = \theta_3(0, \tau), \quad c = \theta_4(0, \tau), \quad (5)$$

we will prove the following

**Proposition 1.5.** *Let  $C_\phi$  be a smooth curve given by the Weierstrass form*

$$\phi(x, y, z) = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3,$$

where  $g_2$  and  $g_3$  belong to a field  $K$ . Then  $\phi$  admits determinantal representations

$$\begin{pmatrix} 2x + tz & y + dz & (3t^2 - g_2)z \\ 0 & x - tz & y - dz \\ z & 0 & -2x - tz \end{pmatrix},$$

with  $t, d \in \overline{K}$  being arbitrary such that  $d^2 = 4t^3 - g_2t - g_3$ . When  $K = \mathbb{C}$ , there is a natural choice for  $t, d$  which produces a determinantal representation for  $\phi$  in terms of theta constants as follows

$$\begin{pmatrix} 2x - \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z & y & -(\frac{\pi}{\omega_1})^4 c^8 z \\ 0 & x + \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z & y \\ z & 0 & -2x + \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z \end{pmatrix},$$

where the even theta constants  $a, b, c$  were defined as in (5).

The first part of this proposition uses the method in [21, Section 2] where the author established similar representations for other type of Weierstrass equations of the form  $y^2z = x(x + \vartheta_1z)(x + \vartheta_2z)$  with some constants  $\vartheta_1, \vartheta_2 \in K$ . The discriminant formula (4) is then a consequence of the second part of this theorem using resultant as in Section 5.

Our goal is to study this phenomena for general smooth cubic curves using determinantal representations. One can actually provide determinantal representations for any non-rational complex plane curve by using a result in [3] as we will see later in Section 6. This deduces in particular the formula of the discriminant of plane cubics by using resultant. Since a cubic curve  $C_\phi$  over  $\mathbb{C}$  always has a flex point, it can be transformed to a Weierstrass form after a linear coordinate change  $M$  (see [8, Section 4.4]). The resulting Weierstrass form is isomorphic to  $\mathbb{C}/\Lambda$  for a unique lattice  $\Lambda$  coming the Weierstrass parametrization. Writing  $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$  for some  $\omega_1, \omega_2 \in \mathbb{C}$  satisfying  $\text{Im}(\omega_2/\omega_1) > 0$ . Denote by  $\tau = \omega_2/\omega_1$ , we will prove the following result.

**Theorem 1.6.** *Let  $C_\phi$  be a smooth plane cubic curve over  $\mathbb{C}$  defined by a cubic form  $\phi$  and  $\Delta_\phi$  be the discriminant of  $\phi$ , we have*

$$\Delta_\phi = \frac{2^{16}}{\det(M)^{12}} \left( \frac{\pi}{\omega_1} \right)^{12} (abc)^8,$$

where  $a, b, c$  were defined as in (5).

This result is known but the above approach with determinantal representations is new. One can compare Theorem 1.6 with Corollary 1.4 for the case when  $n = 3$ . Here we are using a fixed flex point to give a linear transformation from  $C_\phi$  to a Weierstrass form instead of the explicit map  $\varphi_3$ .

So it is natural to link determinantal representations of a smooth plane cubic to a Weierstrass form as the approach using modular forms in Section 1.1. The discriminant formulae will then be predicted. Thus it might be possible to study determinantal representations in more general case using modular forms.

**1.3. Organization of the paper.** We recall the definitions of genus one models as well as their invariants in Section 2. Then we prove Theorem 1.1 and thus Corollary 1.4 in Section 3 by considering the singularity of genus one models. The theory of modular forms will be discussed in Section 4, which will be used to establish Theorem 1.2. After that, we study determinantal representations of Weierstrass cubics and provide a simple proof to Proposition 1.5 in Section 5. Determinantal representations of complex plane curves is presented in Section 6 and we apply it to establish Theorem 1.6 in Section 7.

## 2. MODELS OF GENUS ONE CURVES AND THEIR INVARIANTS

Let  $X_n$  be the set of all genus one models of degree  $n$  over a field  $K$  (see Definition 1.1), we will see later in this section that  $X_n$  is an affine space of dimension 5, 8, 10, 20, 50 for  $n = 1, 2, 3, 4, 5$  respectively. In [12, Section 3], the author defined natural linear algebraic groups  $\mathcal{G}_n$  acting on  $X_n$  ( $n \leq 5$ ), which preserve the solutions of the models. Let  $G_n$  be the commutator subgroups of  $\mathcal{G}_n$  and  $K[X_n]$  be the coordinate ring of  $X_n$ .

**Definition 2.1.** *The ring of invariants of  $X_n$  ( $n \leq 5$ ) over  $K$  is*

$$K[X_n]^{G_n} := \{I \in K[X_n] : I \circ g = I \text{ for all } g \in G_n(\overline{K})\}.$$

*The vector space of invariants of weight  $k$  of  $X_n$  over  $K$  is defined as*

$$K[X_n]_k^{G_n} := \{I \in K[X_n] : I \circ g = (\det g)^k I \text{ for all } g \in G_n(\overline{K})\}.$$

The character  $\det$  on  $\mathcal{G}_n$  is chosen so that we get appropriate weights for the invariants and moreover (see [12, Lemma 4.3])

$$K[X_n]^{G_n} = \bigoplus_{k \geq 0} K[X_n]_k^{G_n}.$$

We now describe in detail affine spaces  $X_n$  for  $n \leq 5$ , their classical invariants and the map  $f_n : C_\phi \rightarrow E_\phi$  ( $n = 2, 3, 4$ ) from a smooth curve defined by a model  $\phi \in X_n^0 := \{\phi \in X_n \mid C_\phi \text{ is smooth}\}$  to its corresponding Jacobian  $E_\phi$ . This map is defined by a divisor  $D$  of degree  $n$  on the curve  $C_\phi$ , which is the intersection of  $C_\phi$  with the hyperplane at infinity (see [1, p. 305]), as  $f_n(P) = nP - D$ . It is given explicitly when  $\text{char}(K) \neq 2, 3$  as below. In addition, we introduce in each degree the natural non-zero regular 1-form  $\omega_\phi$  defined in [12, Section 5.4] for a smooth curve of genus one  $C_\phi$ . This 1-form is useful for relating invariants and modular forms as we will see in Section 4.

**2.1. Models of degree  $n = 1$ .** The space  $X_1$  is of dimension 5 where each model corresponding to (1) can be identify with the point  $(a_1, a_2, a_3, a_4, a_6)$  in  $\mathbb{A}^5$ . Their invariants  $c_4, c_6$  are defined in the usual way as in (2). The natural regular 1-form on a smooth curve  $C_\phi$  defined by model  $\phi$  is:

$$\omega_\phi := \frac{dx}{2y + a_1x + a_3}.$$

**2.2. Models of degree  $n = 2$ .** The equation of a genus one model of degree  $n = 2$  is written as:

$$y^2 + (\alpha_0x^2 + \alpha_1xz + \alpha_2z^2)y = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4. \quad (6)$$

Each model in  $X_2$  corresponds to a point  $(\alpha_0, \alpha_1, \alpha_2, a, b, c, d, e) \in \mathbb{A}^8$  and thus  $\dim(X_2) = 8$ . Moreover, if  $\text{char}(K) \neq 2$  or  $3$ , we can rewrite (6) as:

$$y^2 = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4. \quad (7)$$

As in [1, Section 3.1], the model  $\phi$  defined by (7) has two classical invariants:

$$i = (12ae - 3bd + c^2)/12,$$

$$j = (72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3)/432 \quad (8)$$

and a corresponding Weierstrass equation is:

$$y^2 = 4x^3 - ix - j. \quad (9)$$

The map  $f_2$  from a smooth curve  $C_\phi$  defined by (7) to the Jacobian  $E_\phi$  defined by (9) is given as in [1, (3.3)] by:

$$f_2(x, y, z) = \left( \frac{g(x, z)}{(yz)^2}, \frac{h(x, z)}{(yz)^3} \right),$$

where

$$g(x, z) = \frac{1}{144}(q_{xz}^2 - q_{xx}q_{zz}), \quad h(x, z) = \frac{1}{8} \begin{vmatrix} q_x & q_z \\ g_x & g_z \end{vmatrix}$$

with  $q$  being the binary quartic on the right hand side of (7).

The corresponding results to the generalized equation (6) can be found in [7, p. 766] by completing the square. We define the regular 1-form for a smooth curve  $C_\phi$  defined by a model  $\phi$  corresponding to (6) as:

$$\omega_\phi := \frac{z^2 d(x/z)}{2y + p(x, z)}.$$

Here  $p(x, z) = \alpha_0x^2 + \alpha_1xz + \alpha_2z^2$ .

**2.3. Models of degree  $n = 3$ .** In case  $n = 3$ , a genus one model  $\phi$  is a ternary cubic:

$$ax^3 + by^3 + cz^3 + a_2x^2y + a_3x^2z + b_1xy^2 + b_3y^2z + c_1xz^2 + c_2yz^2 + mxyz \quad (10)$$

with two classical invariants  $S, T$  defined in [1, p. 309-310] and we can see that  $\dim(X_3) = 10$ . A corresponding Weierstrass equation is also given there by:

$$y^2 = 4x^3 + 108Sx - 27T. \quad (11)$$

The map  $f_3$  from a smooth curve  $C_\phi$  defined by the model (10) to the Jacobian  $E_\phi$  defined by (11) is given as in [1, (3.9)] by:

$$f_3(x, y, z) = \left( \frac{\Theta(x, y, z)}{H(x, y, z)^2}, \frac{J(x, y, z)}{H(x, y, z)^3} \right),$$

where

$$H = \frac{1}{216} \begin{vmatrix} \phi_{xx} & \phi_{xy} & \phi_{xz} \\ \phi_{yx} & \phi_{yy} & \phi_{yz} \\ \phi_{zx} & \phi_{zy} & \phi_{zz} \end{vmatrix}, \quad J = -\frac{1}{9} \left| \frac{\partial(\phi, H, \Theta)}{\partial(x, y, z)} \right|$$

and  $\Theta$  is the covariant defined in [1, p. 308]. The regular 1-form on the smooth curve  $C_\phi$  is defined as

$$\omega_\phi := \frac{x^2 d(y/x)}{\partial\phi/\partial z}.$$

**2.4. Models of degree  $n = 4$ .** We relate to the case  $n = 2$  as follows: if the model  $\phi$  is given by a pair of quadrics  $q_1, q_2 \in K[x_0, x_1, x_2, x_3]$  (hence  $\dim(X_4) = 20$ ), we can write  $q_1 = \bar{x}A\bar{x}^T$  and  $q_2 = \bar{x}B\bar{x}^T$  for two symmetric  $4 \times 4$  matrices  $A, B$  with  $\bar{x} = (x_0, x_1, x_2, x_3)$ . The invariants of  $\phi$  are then defined by the invariants of the quartic:

$$\det(xA + zB) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4 \quad (12)$$

as in the case  $n = 2$ . A corresponding Weierstrass equation is thus given in the form (9) with  $i, j$  defined by the coefficients of the model (12) as in (8). The explicit map  $f_4$  from a smooth curve defined by (12) to the Jacobian is given in [1, (3.12)] as:

$$f_4(x_0, x_1, x_2, x_3) = \left( \frac{g}{J^2}, \frac{h}{J^3} \right),$$

where  $g, h, J$  are defined as in [1, Section 3.3]. The regular 1-form for the smooth curve  $C_\phi$  is defined as:

$$\omega_\phi := \frac{x_0^2 d(x_1/x_0)}{(\partial q_1/\partial x_3)(\partial q_2/\partial x_2) - (\partial q_1/\partial x_2)(\partial q_2/\partial x_3)}.$$

**2.5. Models of degree  $n = 5$ .** A genus one model of degree  $n = 5$  is a  $5 \times 5$  alternating matrix of linear forms in five variables. The equations defined by this model are the  $4 \times 4$  Pfaffians of the matrix. Here a square matrix is called alternating if it is skew-symmetric and all of its diagonal entries are zero. The reader can have a look at [12, Section 5.2] and [14] for reference. We can check that  $\dim(X_5) = 50$ . Fisher [12, p. 770] also defines a regular 1-form to the case  $n = 5$  when  $\text{char}(K) \neq 2$ .



Observe that if  $I$  is an invariant of weight  $k$  then so is  $\lambda I$  for any constant  $\lambda \in K^*$ . We want to normalize the invariants so that they have appropriate formulae in any characteristic.

**2.6. Normalized invariants.** The author in [12, Theorem 10.2] proves that  $K[X_n]^{G_n}$  is isomorphic to  $K[X_1]^{G_1}$  ( $n \leq 5$ ) in any characteristic. This extends the invariants  $c_4, c_6, \Delta \in \mathbb{Z}[X_1]$  of  $K[X_1]^{G_1}$  defined in (2) to the corresponding ones in  $K[X_n]^{G_n}$  ( $n \leq 5$ ) denoted again by  $c_4, c_6, \Delta$ . In fact, we have (see [12, Lemma 4.15 and remark 4.16])

**Lemma 2.2.** *The invariants  $c_4, c_6$  and  $\Delta$  are primitive polynomial in  $\mathbb{Z}[X_n]$  for any  $n \leq 5$ .*

Thus it is possible to normalize these invariants up to sign. Furthermore, the two invariants  $c_4, c_6 \in K[X_n]^{G_n}$  ( $n \leq 5$ ) constructed above define the suitable discriminant  $\Delta = (c_4^3 - c_6^2)/1728$  in the following way:

**Definition/Lemma 2.3.** *Let  $R$  be a unital commutative ring and  $C_\phi$  be a curve over  $R$  defined by some genus one model  $\phi$  of degree  $n \leq 5$ . There exists the discriminant  $\Delta$ , which is a universal polynomial with integer coefficients, is defined such that  $\Delta_\phi \in R^*$  if and only if  $C_\phi$  is non-singular over  $R$ . Here  $R^*$  is the group of units of  $R$ .*

*Proof.* Fisher [12, Theorem 4.4] shows the properties of the discriminant of genus one models of degree  $n \leq 5$  over a field  $K$ , but it is indeed equivalent to the Definition/Lemma 2.3.  $\square$

He also proves there that if  $\text{char}(K) \neq 2$  or  $3$ , then the invariants  $c_4(\phi), c_6(\phi)$  provide for the smooth genus one curve  $C_\phi$  defined by a model  $\phi$  of degree  $n \leq 5$  the Jacobian

$$y^2 = x^3 - 27c_4(\phi)x - 54c_6(\phi).$$

The main ingredient of the proof of the above result is that, a pair  $(C_\phi, \omega_\phi)$  of a model  $\phi \in X_n^0$  is isomorphic to a pair of the form given in Section 2.1. Then the invariants  $c_4(\phi), c_6(\phi)$  of  $\phi$  are determined by the ones of that pair as in (2) (see [12, Definition 2.1 and Proposition 5.23]).

Then, in [12, Section 7], he gives the normalized formulae to the invariants  $c_4, c_6$  of genus one models of degrees  $n = 2, 3, 4$ . More precisely, in the case  $n = 2$ , the normalized invariants of the model corresponding to (7) are

$$\begin{aligned} c_4 &= 2^4(12ae - 3bd + c^2), \\ c_6 &= 2^5(72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3). \end{aligned} \tag{13}$$

In comparison with the classical case (8), we have  $c_4 = 2^6 3i$  and  $c_6 = 2^9 3^3 j$ . The normalized invariants of the generalized model corresponding to (6) can be found in [7, p. 766] by reducing to the form (7) from a completing square.

When  $n = 3$ , the normalized invariants of the model (10) are

$$\begin{aligned} c_4 &= -216abcm + 144abc_1c_2 + 144acb_1b_3 + \dots - 8a_3b_3m^2 + 16b_1^2c_1^2 - 8b_1c_1m^2 + m^4, \\ c_6 &= 5832a^2b^2c^2 - 3888a^2bcb_3c_2 + 864a^2bc_2^3 + \dots + 64b_1^3c_1^3 - 48b_1^2c_1^2m^2 + 12b_1c_1m^4 - m^6. \end{aligned} \quad (14)$$

We have  $c_4 = -2^4 3^4 S$ ,  $c_6 = 2^3 3^6 T$  in comparing with the classical invariants  $S, T$  defined in [1, p. 309-310].

When  $n = 4$ , we relate to the quartic as in (12) and then the invariants are

$$\begin{aligned} c_4 &= 12ae - 3bd + c^2, \\ c_6 &= \frac{1}{2}(72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3). \end{aligned} \quad (15)$$

Thus in comparing with the classical case, we have  $c_4 = 2^{10} 3i$  and  $c_6 = 2^{15} 3^3 j$ .

In the case of plane cubics, these invariants were normalized before in [2, p. 367-368] where the authors explicitly provide the corresponding Weierstrass equations in any characteristic. Strictly speaking, they associate to a ternary cubic  $\phi$  a Weierstrass form  $\phi^*$  associated to (1) with the coefficients determined by the coefficients of  $\phi$ . Then they observe that  $(\phi^*)^* = \phi^*$  and thus naturally define:

$$c_4(\phi) := c_4(\phi^*), c_6(\phi) := c_6(\phi^*), \Delta_\phi := \Delta_{\phi^*}.$$

In Section 4, we will provide a new way for expressing these normalized invariants using modular forms. Before that, however, we will establish the normalized discriminants of genus one models directly by using the singularities of the models in the next section.

### 3. DISCRIMINANT OF GENUS ONE CURVES AND ITS JACOBIANS

The goal of this section is to prove Theorem 1.1. Fix  $n \leq 5$ , let  $X_n$  be the affine space of all genus one models  $\phi$  of degree  $n$ , let  $W$  be the affine space of Weierstrass forms  $y^2 - 4x^3 + g_2x + g_3$  if  $\text{char}(K) \neq 2, 3$  and  $W = X_1$  otherwise. We define the map  $\varphi_n : X_n \rightarrow W$  based on the discussion about the map  $f_n$  in Section 2 for  $n = 2, 3, 4$ . More precisely, for any smooth model  $\phi \in X_n^0$ , we have a map  $f_n : C_\phi \rightarrow E_\phi$  from the smooth curve  $C_\phi$  to its Jacobian  $E_\phi$  coming from a divisor of degree  $n$ . We define the image  $\varphi_n(\phi)$  of  $\phi$  to be the model in  $W$  defining  $E_\phi$ . This gives us a map  $X_n^0 \rightarrow W$  which extends uniquely to a map  $\varphi_n : X_n \rightarrow W$ .

The map  $\varphi_n$  is given explicitly in case  $\text{char}(K) \neq 2$  or  $3$ , which also applies to singular models as follows. The map  $\varphi_2$  sends a model  $\phi$  of degree 2 corresponding to (6) to the Weierstrass form defined by (9) of the model corresponding to (7) obtained from  $\phi$  after a completing square. The map  $\varphi_3$  sends a model  $\phi$  of degree 3 of the form

(10) to the Weierstrass form defined by (11). The map  $\varphi_4$  is defined by relating to the case  $n = 2$ . More precisely,  $\varphi_4$  sends a model  $\phi$  of degree 4 given by a pair of quadrics  $(q_1, q_2)$  to the Weierstrass form defined by (9) obtained from the quartic (12) as in the case  $n = 2$ . We denote by  $E_\phi$  the curve given by the corresponding Weierstrass form  $\varphi_n(\phi) \in W$  of a model  $\phi \in X_n$ .

This map sends non-singular curves to non-singular curves and singular curves to singular curves when  $\text{char}(K) \neq 2, 3$ . We know that there exist discriminants (see Definition/Lemma 2.3)  $\Delta_{X_n}$  in  $X_n$  and  $\Delta_W$  in  $W$  parametrizing singular curves and they are both geometrically irreducible polynomials (see [12, Proposition 4.5]). The discriminant  $\Delta_\phi$  of  $\phi$  is determined by evaluating  $\Delta_{X_n}$  at coefficients of the model  $\phi$ . i.e.,  $\Delta_\phi = \Delta_{X_n}(\phi)$ .

We denote by  $V(p)$  the set of points in  $\mathbb{P}_K^m$  vanished at  $p$  for any homogeneous polynomial  $p \in K[x_0, \dots, x_m]$ . Observe that  $\Delta_{X_n}$  and the pull back  $\varphi_n^*(\Delta_W)$  of  $\Delta_W$  have the same vanishing property:

$$V(\Delta_{X_n}) = V(\varphi_n^*(\Delta_W)), \quad (16)$$

where  $\varphi_n^*$  is defined such that  $\varphi_n^*(\Delta_W)(\phi) = \Delta_W(\varphi_n(\phi))$  for any model  $\phi$ . We have the following:

**Proposition 3.1.** *For  $n \leq 4$ , there exists a constant  $c \in K^*$  such that*

$$\Delta_{X_n} = c\varphi_n^*(\Delta_W).$$

*Proof.* For an ideal  $J \subset K[x_0, \dots, x_m]$ , we denote by  $\sqrt{J}$  the radical ideal of  $J$  over  $\overline{K}$ . From (16) and Hilbert's Nullstellensatz, we obtain that  $\sqrt{(\Delta_{X_n})} = \sqrt{(\varphi_n^*(\Delta_W))}$ . Note that  $\Delta_{X_n}$  is geometrically irreducible as mentioned above. Then  $\sqrt{(\Delta_{X_n})} = (\Delta_{X_n})$  (which is an ideal over  $\overline{K}$ ) and thus there exist constants  $k \in \mathbb{Z}$  and  $c \in \overline{K}^*$  such that  $\Delta_{X_n}^k = c\varphi_n^*(\Delta_W)$ . Since both  $\Delta_{X_n}^k$  and  $\varphi_n^*(\Delta_W)$  are defined over  $K$ , we conclude that  $c \in K^*$ .

We will prove that  $k = 1$  by comparing the degrees of  $\Delta_{X_n}$  and  $\varphi_n^*(\Delta_W)$ . We first consider the case  $n = 3$ . By [5, Example 1.8], we know that  $\Delta_{X_n}$  is a homogeneous polynomial of degree 12 with respect to the coefficients of the plane cubics. Besides,  $\varphi_3^*(\Delta_W)$  is also of degree 12 in the coefficients of the cubics so that it has the same degree with  $\Delta_{X_n}$ .

If  $n = 4$ , by [5, Example 1.10] we see that  $\Delta_{X_n}$  is a homogeneous polynomial of degree 24 in the coefficients of the two quadratic forms defining the models. The degree of  $\varphi_4^*(\Delta_W)$  is also 24.

The case  $n = 2$  is different since the model  $y^2 + p(x, z)y = q(x, z)$  is no longer homogeneous. Here  $p, q$  are homogeneous polynomials of degrees 2, 4 respectively. If  $\text{char}(K) \neq 2$ , this model can be brought to  $y^2 = h(x, z)$  by completing square, where

$h = \frac{p^2}{4} + q$ . The singular locus of this latter model is  $\{2y = h_x = h_z = 0\}$ , which is equal to  $\{h_x = h_z = 0\}$  if  $\text{char}(K) \neq 2$ . Consequently, the discriminant of the model  $(p, q)$  above is the discriminant of the quartic  $h$  from Definition/Lemma 2.3 (up to some power of 2). Hence, by [5, Example 1.8] again, we know that  $\Delta_{X_n}(p, q)$  is of degree 6 with respect to the coefficients of  $h$ . We can check that  $\varphi_2^*(\Delta_W)(p, q)$  is of degree 6 in terms of coefficients of  $h$  as well. Thus  $k = 1$  for  $n = 2, 3, 4$ .  $\square$

We actually have more information about the constant  $c$  by looking at models with integer coefficients.

**Proposition 3.2.** *For  $n = 2, 3, 4$ , the constant  $c$  in Proposition 3.1 can be expressed as  $\pm 2^a$  for some  $a \in \mathbb{Z}$ . Moreover,*

$$\begin{cases} a = 0, & \text{If } n = 2; \\ a = -12, & \text{If } n = 3; \\ a = 12, & \text{If } n = 4. \end{cases}$$

We will see later in Theorem 4.10 that one can exclude the minus sign of the constant  $c$  above and thus obtain Theorem 1.1.

*Proof.* We consider the curve  $C_\phi$  defined by a model  $\phi$  with integer coefficients so that we can make use of reduction modulo prime numbers. Note that  $\Delta_\phi \in \mathbb{Z}$  by Lemma 2.2. Since the map  $\varphi_n$  sends non-singular curves to non-singular curves over characteristics not 2 and 3, the constant  $c$  is of the form  $\pm 2^a 3^b$  for some integers  $a, b$ . To compute the powers of 2 and 3, we need to compare  $\Delta_\phi$  and  $\Delta_{\varphi_n(\phi)}$  over  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  (see Definition/Lemma 2.3). Here  $\Delta_{\varphi_n(\phi)} = \Delta_W(\varphi_n(\phi))$  is the discriminant of the Weierstrass form  $\varphi_n(\phi)$  of  $\phi$ . Since  $\Delta_\phi = c\Delta_{\varphi_n(\phi)}$ , we get the following  $p$ -adic valuation identity for a prime number  $p$ :

$$v_p(\Delta_\phi) = v_p(c) + v_p(\Delta_{\varphi_n(\phi)}). \quad (17)$$

Suppose that  $C_\phi$  is non-singular over  $\mathbb{F}_2$  and  $\mathbb{F}_3$ , then  $v_2(\Delta_\phi) = v_3(\Delta_\phi) = 0$  and we get from (17) that  $v_2(c) = -v_2(\Delta_{\varphi_n(\phi)})$ ,  $v_3(c) = -v_3(\Delta_{\varphi_n(\phi)})$ . So to find  $c$ , we just need to compute  $\Delta_{\varphi_n(\phi)}$  in some special case in which  $C_\phi$  is non-singular over  $\mathbb{F}_2$  and  $\mathbb{F}_3$ .

In the case  $n = 2$ , we consider the genus one model  $\phi$  with equation

$$y^2 + yz^2 = x^4 + x^3z + x^2z^2.$$

The corresponding Weierstrass equation of  $\varphi_2(\phi)$  is:

$$y^2 = 4x^3 - \frac{1}{3}x - \frac{37}{1728}$$

and  $\Delta_{\varphi_2(\phi)} = 101$ . The model  $\phi$  is non-singular over both  $\mathbb{F}_2$  and  $\mathbb{F}_3$ . Hence  $v_2(c) = 0$ ,  $v_3(c) = 0$  and thus  $a = 0, b = 0$ .

If  $n = 3$ , we consider the curve  $C_\phi$  given by  $y^2z + yz^2 - x^3 = 0$  with the corresponding equation of  $\varphi_3(\phi) : y^2 = 4x^3 + 1$  and we have  $\Delta_{\varphi_3(\phi)} = -2^{12}3^3$ . The curve  $C_\phi$  is non-singular over  $\mathbb{F}_2$  and thus  $v_2(c) = -12$  or  $a = -12$ . To compute the power of 3, we consider the following curve  $y^2z - x^3 - xz^2 = 0$  which is non-singular over  $\mathbb{F}_3$  with the corresponding equation of  $\varphi_3(\phi) : y^2 = 4x^3 + 4x$  and  $\Delta_{\varphi_3(\phi)} = -2^{18}$ . Therefore, we have  $v_3(c) = 0$  or  $b = 0$ .

When  $n = 4$ , we consider the curve  $C_\phi$  given by the following complete intersection of two quadratic forms

$$\begin{cases} x_0x_1 + x_0x_2 + x_2x_3 = 0 \\ x_0x_3 + x_1x_2 + x_1x_3 = 0 \end{cases}$$

and the corresponding Weierstrass equation

$$y^2 = 4x^3 - \frac{1}{2^{10}3}x + \frac{161}{2^{15}3^3}$$

computed by  $\varphi_4$  with  $\Delta_{\varphi_4(\phi)} = -3.5/2^{12}$ . It is possible to check that  $C_\phi$  is non-singular over  $\mathbb{F}_2$ . This implies that  $v_2(c) = 12$  and hence  $a = 12$ . Observe that this curve is singular over  $\mathbb{F}_3$  since  $(1, 1, 1, 1)$  is a singular point modulo 3. So to compute the power of 3, we need to look at another example. For instance, we consider the following complete intersection which is non-singular over  $\mathbb{F}_3$

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + 3x_3^2 = 0 \\ x_0^2 + 2x_1^2 + 3x_2^2 + 5x_3^2 = 0 \end{cases}$$

with the corresponding equation of  $\varphi_4(\phi) : y^2 = 4x^3 - x$  and we obtain in this case that  $\Delta_{\varphi_4(\phi)} = 2^{12}$ . This means that  $v_3(c) = 0$  and thus  $b = 0$ . This completes the proof of Proposition 3.2.  $\square$

#### 4. INVARIANTS AND MODULAR FORMS

We now in this section study all invariants of genus one models. To do this, we first give a brief introduction to the theory of modular forms and the connection to invariants. The goal is to establish Theorem 1.2 by proving Theorem 4.10, which is the main result of this section.

**4.1. Weakly holomorphic and geometric modular forms.** A weakly holomorphic modular form  $F$  of weight  $k \in \mathbb{Z}$  is a holomorphic function on the upper half-plane  $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}\tau > 0\}$ , that is meromorphic at  $\infty$  and satisfies the equation

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k F(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

where

$$SL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Denote by  $M_k^!(\mathbb{C})$  the space of weakly holomorphic modular forms of weight  $k$  and  $M^!(\mathbb{C})$  the graded algebra

$$M^!(\mathbb{C}) := \bigoplus_{k \in \mathbb{Z}} M_k^!(\mathbb{C}).$$

$F$  is called holomorphic if it is holomorphic at  $\infty$ , i.e.,  $F$  has a Fourier expansion

$$F(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$$

which is absolutely convergent for each  $\tau \in \mathbb{H}$ . We denote by  $M_k(\mathbb{C})$  the space of holomorphic modular forms of weight  $k$  and  $M(\mathbb{C})$  the graded algebra

$$M(\mathbb{C}) := \bigoplus_{k \geq 0} M_k(\mathbb{C}).$$

One of the most important examples of holomorphic modular forms is the Eisenstein series  $G_{2k}$ , which is of weight  $2k$ , is defined for an integer  $k \geq 2$  as

$$G_{2k} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m + n\tau)^{2k}}. \quad (18)$$

We usually use the following normalized notation of the Eisenstein series

$$E_{2k} := \frac{G_{2k}}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{+\infty} \sigma_{2k-1}(n) q^n, \quad (19)$$

where  $\zeta$  is the Riemann zeta function,  $B_{2k}$  are the Bernoulli numbers,  $\sigma$  is the divisor sum function and  $q = e^{2\pi i \tau}$ .

Next step is to follow [18, p. 9,10] to introduce the notion of geometric modular forms. Here an elliptic curve  $E$  over a scheme  $S$  is a proper smooth morphism  $\pi : E \rightarrow S$  whose generic fibers are connected smooth curves of genus one together with a section  $e : S \rightarrow E$ .

**Definition 4.1.** *A geometric modular form of weight  $k \in \mathbb{Z}$  over a scheme  $S$  is a rule  $\mathcal{F}$  which assigns to every pair  $(E/R, \omega)$  of an elliptic curve  $\pi : E \rightarrow R$  over  $S$  and a basis  $\omega$  of  $\pi_* \Omega_{E/R}^1$  an element  $\mathcal{F}(E/R, \omega) \in R$  such that*

- a)  $\mathcal{F}(E/R, \omega)$  depends only on the  $R$ -isomorphism class of the pair  $(E/R, \omega)$ .
- b) For any  $\lambda \in R^*$  we have  $\mathcal{F}(E, \lambda\omega) = \lambda^{-k} \mathcal{F}(E, \omega)$ .
- c)  $\mathcal{F}(E'/R', \omega_{R'}) = \psi(\mathcal{F}(E/R, \omega))$  for any morphism  $\psi : R \rightarrow R'$ , i.e.,  $\mathcal{F}$  commutes with arbitrary base change. Here  $(E'/R', \omega_{R'})$  is the base change of  $(E/R, \omega)$  along  $\psi$ .

We adopt the same definition if we only assume that  $E/R$  is a smooth genus one curve over  $R$  by the following lemma, which is surely known to the experts but the author was unable to locate it in the literature.

**Lemma 4.2.** *There is a natural correspondence between:*

- *Geometric modular forms of weight  $k$  for elliptic curves over a scheme  $S$ .*
- *Geometric modular forms of weight  $k$  for smooth genus one curves over a scheme  $S$ .*

*Proof.* Suppose first that we are given a geometric modular form for curves of genus one,  $\mathcal{F}$ . An elliptic curve  $E/R$  is a pair  $(C, e)$  where  $C$  is a smooth genus one curve  $C$  over  $R$  and a section of  $e : \text{Spec}(R) \rightarrow C$ . We can simply forget the section and set  $\mathcal{F}(E/R, \omega) := \mathcal{F}(C/R, \omega)$ . This will satisfy all the properties in the definition because  $\mathcal{F}(C/R, \omega)$  does.

We have to show the converse, and suppose we are given a geometric modular form  $\mathcal{F}$  for elliptic curves. Locally for the étale topology (see [16, 17.16.3 (ii)]), there are étale ring extensions  $\psi_i : R \rightarrow R_i$  such that  $C_i := C \otimes_R R_i$  admits a section,  $e$ , over  $R_i$ , giving an elliptic curve  $E_i/R_i$ . We argue that  $\mathcal{F}(E_i/R_i, \omega_i)$  is independent of the choice of section. One can compare  $e$  with another choice of section  $P_i : \text{Spec}(R_i) \rightarrow C_i$  by the  $R_i$ -isomorphism given by translation by  $P_i$ ,  $\tau_{P_i} : (C_i, e_i) \simeq (C_i, P_i)$ . This is an isomorphism of elliptic curves. Since the differential  $\omega_i$  is invariant under  $\tau_{P_i}$ , the property *a*) in Definition 4.1 shows that  $\mathcal{F}((C_i, e), \omega_i) = \mathcal{F}((C_i, P_i), \omega_i)$ .

We then set  $\alpha_i := \mathcal{F}(E_i/R_i, \omega_i) \in R_i$ , independent of any choice of section. Let  $R_j$  be another (local) ring extension such that  $C_j/R_j$  admits a section, and denote by  $\psi_{ij} : R_i \rightarrow R_{ij} = R_i \otimes_R R_j$  the natural map. Then  $\psi_{ji}(\alpha_j) = \alpha_{ij} = \psi_{ij}(\alpha_i)$  by the property *c*) of Definition 4.1. By étale descent we obtain an element  $\alpha \in R$  such that for all the maps  $\psi_i : R \rightarrow R_i$ ,  $\psi_i(\alpha) = \alpha_i$ . It is a straightforward verification that  $\mathcal{F}(C/R, \omega) = \alpha$  satisfies the definition of a geometric modular form of a genus one curve.  $\square$

Denote by  $\mathcal{M}_k^!(R)$  the  $R$ -module of geometric modular forms of weight  $k$  over a ring  $R$  and  $\mathcal{M}^!(R)$  the graded algebra

$$\mathcal{M}^!(R) := \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k^!(R).$$

As in [18, p. 10], a geometric modular form  $\mathcal{F}$  has its  $q$ -expansion as an element of  $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} R$  obtained by evaluating on the pair  $(\text{Tate}(q), \omega)_R$  consisting of the Tate curve and its canonical differential.  $\mathcal{F}$  is called holomorphic if it is holomorphic at  $\infty$ , i.e., its  $q$ -expansion lies in  $\mathbb{Z}[[q]] \otimes_{\mathbb{Z}} R$ .



We denote by  $\mathcal{M}_k(R)$  the  $R$ -module of holomorphic geometric modular forms of weight  $k$  over a ring  $R$  and  $\mathcal{M}(R)$  the graded algebra

$$\mathcal{M}(R) := \bigoplus_{k \geq 0} \mathcal{M}_k(R).$$

We have in addition the relation

$$\mathcal{M}^!(R) = \mathcal{M}(R)[\mathcal{D}^{-1}], \quad (20)$$

where  $\mathcal{D}$  is the cusp form of weight 12 defined below.

It turns out that we can identify weakly holomorphic and geometric modular forms over  $\mathbb{C}$  from the following discussion in [18, p. 91]. Let  $C_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  for any  $\tau \in \mathbb{H}$ . For any geometric modular form  $\mathcal{F} \in \mathcal{M}_k^!(\mathbb{C})$ , we can define the corresponding weakly holomorphic modular form of the same weight

$$F(\tau) = \mathcal{F}(C_\tau, 2\pi i \, dz) \quad (21)$$

with  $dz$  being the canonical differential on  $\mathbb{C}$ . Then the map  $\mathcal{F} \mapsto F$  is an isomorphism  $\mathcal{M}_k^!(\mathbb{C}) \cong M_k^!(\mathbb{C})$ .

Observe that for  $k = 2, 3$  the quotient  $4k/B_{2k} \in \mathbb{Z}$  and thus the Eisenstein series  $E_4, E_6$  in (19) are defined over  $\mathbb{Z}$ . By the  $q$ -expansion principle as in [18, Corollary 1.9.1], the corresponding geometric modular forms  $\mathcal{E}_{2k}$  of  $E_{2k}$  of weight  $2k$  ( $k = 2, 3$ ) are also defined over  $\mathbb{Z}$ .

It is known that the ring of holomorphic geometric modular forms  $\mathcal{M}(\mathbb{Z})$  over  $\mathbb{Z}$  is generated by  $\mathcal{E}_4, \mathcal{E}_6$  and the cusp form  $\mathcal{D}$  satisfying  $1728\mathcal{D} = \mathcal{E}_4^3 - \mathcal{E}_6^2$ . More precisely, we have (see [9, Proposition 6.1])

$$\mathcal{M}(\mathbb{Z}) \cong \mathbb{Z}[\mathcal{E}_4, \mathcal{E}_6, \mathcal{D}]/(\mathcal{E}_4^3 - \mathcal{E}_6^2 - 1728\mathcal{D}).$$

If  $\text{char}(K) \neq 2$  or  $3$ , then  $1728$  is invertible over  $K$  and thus  $\mathcal{M}(K) = K[\mathcal{E}_4, \mathcal{E}_6]$ . In this case,  $\mathcal{E}_4$  and  $\mathcal{E}_6$  are algebraically independent since so are  $E_4$  and  $E_6$ .

We have a type of geometric modular forms in any characteristic  $p$  called the Hasse invariants defined for instance in [18] (p. 29). For any prime number  $p$ , the Hasse invariant  $\mathcal{A}_p$  is a geometric modular form over  $\mathbb{F}_p$  of weight  $p - 1$ , which satisfy a certain property. The Hasse invariant  $\mathcal{A}_p$  has  $q$ -expansion equal to 1 in  $\mathbb{F}_p[[q]]$ . For any prime  $p > 3$ , we have  $\mathcal{A}_p = \mathcal{E}_{p-1} \pmod{p}$  since they are both geometric modular forms of the same weight  $p - 1$  with the same  $q$ -expansions (see [18] (p. 30)).

The structure of the graded ring of holomorphic geometric modular forms can be summarized from Propositions 6.1, 6.2, Remark 6.3 and the formula (8.4) in [9] as follows:



**Proposition 4.3.** *The graded ring  $\mathcal{M}(K)$  of holomorphic geometric modular forms over a field  $K$  is*

$$\begin{cases} K[\mathcal{E}_4, \mathcal{E}_6], & \text{if } \text{char}(K) \neq 2, 3; \\ K[\mathcal{A}_2, \mathcal{D}], \mathcal{E}_4 = \mathcal{A}_2^4 \text{ and } \mathcal{E}_6 = -\mathcal{A}_2^6, & \text{if } \text{char}(K) = 2; \\ K[\mathcal{A}_3, \mathcal{D}], \mathcal{E}_4 = \mathcal{A}_3^2 \text{ and } \mathcal{E}_6 = -\mathcal{A}_3^3, & \text{if } \text{char}(K) = 3. \end{cases}$$

**4.2. Invariants and modular forms.** Similarly, the structure of the graded ring of invariants of  $X_n$  can be summarized from [12, Theorem 4.4, Lemma 10.1, Theorem 10.2] as below. Here  $c_4, c_6$  are the usual invariants defined in Section 2 and  $a_1, b_2$  are the invariants of weight 1, 2 respectively defined as in [12, Theorem 10.2].

**Proposition 4.4.** *The ring of invariants  $K[X_n]^{G_n}$  of  $X_n$  ( $n \leq 5$ ) over a field  $K$  is*

$$\begin{cases} K[c_4, c_6], & \text{if } \text{char}(K) \neq 2, 3; \\ K[a_1, \Delta], & \text{if } \text{char}(K) = 2; \\ K[b_2, \Delta], & \text{if } \text{char}(K) = 3. \end{cases}$$

The algebraic independence of the invariants  $c_4$  and  $c_6$  if  $\text{char}(K) \neq 2$  or  $3$ ,  $a_1$  and  $\Delta$  if  $\text{char}(K) = 2$ ,  $b_2$  and  $\Delta$  if  $\text{char}(K) = 3$  is clear in case  $n = 1$ . Thus they are algebraically independent for all  $n \leq 5$ .

We can link modular forms to invariants. To see this, we first recall a result from [12, Proposition 5.19]. Here the regular 1-form  $\omega_\phi$  of a model  $\phi \in X_n^0$  is defined as in Section 2.

**Lemma 4.5.** *Let  $C_\phi, C_{\phi'}$  be smooth curves of genus one over a field  $K$  corresponding to the models  $\phi, \phi' \in X_n^0$  respectively ( $n \leq 4$ ). Suppose  $\phi' = g\phi$  for some  $g \in \mathcal{G}_n$ , then the isomorphism  $\varphi : C_{\phi'} \rightarrow C_\phi$  determined by  $g$  satisfies  $\varphi^*\omega_\phi = (\det g)\omega_{\phi'}$ . This statement holds to the case  $n = 5$  providing that  $\text{char}(K) \neq 2$ .*

The author in [12] provides an explicit proof for  $n \leq 5$  corresponding to the explicit linear algebraic groups  $\mathcal{G}_n$  acting on  $X_n$ . We observe from this lemma that for any  $g \in \mathcal{G}_n$  and any genus one curves  $C_\phi, C_{\phi'}$  defined by models  $\phi, \phi' = g\phi$  in  $X_n^0$ , we have for a geometric modular form  $\mathcal{F}$  of weight  $k$

$$\mathcal{F}(C_\phi, \omega_\phi) = \mathcal{F}(C_{\phi'}, \varphi^*\omega_\phi) = \mathcal{F}(C_{\phi'}, (\det g)\omega_{\phi'}) = (\det g)^{-k} \mathcal{F}(C_{\phi'}, \omega_{\phi'})$$

or

$$\mathcal{F}(C_{\phi'}, \omega_{\phi'}) = (\det g)^k \mathcal{F}(C_\phi, \omega_\phi).$$

We have thus proved

**Proposition 4.6.** *For  $n \leq 4$ , a geometric modular form  $\mathcal{F}$  over a field  $K$  defines an invariant of the same weight  $I_{\mathcal{F}}$  over  $K$  of  $X_n^0$  such that  $I_{\mathcal{F}}(\phi) = \mathcal{F}(C_\phi, \omega_\phi)$  for any  $\phi \in X_n^0$ . This statement holds to the case  $n = 5$  providing that  $\text{char}(K) \neq 2$ .*

This fact over the complex numbers can also be seen directly from holomorphic modular forms. For a smooth curve  $C$  given in the Weierstrass form  $\phi$  such that  $C(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  for some  $\tau \in \mathbb{H}$ . Then  $\phi$  is of the form  $y^2 - 4x^3 + g_2x + g_3$ , whose the two coefficients  $g_2, g_3$  is written as

$$g_2 = 60G_4, g_3 = 140G_6 \quad (22)$$

with  $G_4, G_6$  defined in (18). The normalized invariants of the model  $\phi$  are (see Remark 1.3)

$$c_4(\phi) = 2^6 3g_2, c_6(\phi) = 2^9 3^3 g_3. \quad (23)$$

The following identities are then a consequence of (19), (22), (23) and the special values of zeta function  $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$

$$c_4(\phi) = (4\pi)^4 E_4(\tau), c_6(\phi) = (4\pi)^6 E_6(\tau). \quad (24)$$

It is then possible to compare the invariant  $I_{\mathcal{F}}$  in case  $\mathcal{F} = \mathcal{E}_4$  and  $\mathcal{E}_6$  with the normalized ones  $c_4, c_6$  as follows

**Lemma 4.7.** *We have the following identities for any smooth model  $\phi \in X_n^0$  ( $n \leq 4$ ) over a field  $K$*

$$I_{\mathcal{E}_4}(\phi) = c_4(\phi), I_{\mathcal{E}_6}(\phi) = -c_6(\phi).$$

*This statement holds to the case  $n = 5$  providing that  $\text{char}(K) \neq 2$ .*

*Proof.* Since both  $I_{\mathcal{E}_{2k}}(\phi)$  and  $c_{2k}(\phi)$  ( $k = 2, 3$ ) depend only on the  $K$ -isomorphism class of the pair  $(C_\phi, \omega_\phi)$  (see Definition 4.1 and [12, Definition 2.1 and Proposition 5.23]), it is enough to consider the case  $n = 1$ . We know that both  $I_{\mathcal{E}_4}$  and  $c_4$  are defined over  $\mathbb{Z}$ . Moreover,  $c_4$  is a primitive polynomial. There exists thus a constant  $\alpha \in \mathbb{Z}$  such that  $I_{\mathcal{E}_4}(\phi) = \alpha c_4(\phi)$  for any  $\phi \in X_1^0$ . Consider a model  $\phi$  with coefficients in  $\mathbb{Z} \subset \mathbb{C}$ , we obtain from (21) and (24) that  $I_{\mathcal{E}_4}(\phi) = c_4(\phi)$  and hence  $\alpha = 1$ . Here we are using the regular 1-form  $\omega_\phi = dx/2y$  while the differential  $dz$  in (21) is equal to  $dx/y$ . Similarly, we have  $I_{\mathcal{E}_6}(\phi) = -c_6(\phi)$  for any  $\phi \in X_1^0$ .  $\square$

One can look closer at the connection between invariants and geometric modular forms. The author in [9, Propositions 6.1, 6.2 and Remark 6.3] proved that  $\mathcal{M}(K)$  is isomorphic to  $K[X_1]^{G_1}$  over any field  $K$ . We want to get a similar phenomena when replacing  $X_1$  by  $X_n$  for any  $n \leq 5$ . Denote by  $K[X_n^0]^{G_n}$  the ring of invariants of  $X_n^0$  defined in the same way as in Definition 2.1, i.e.,

$$K[X_n^0]^{G_n} := \{I \in K[X_n^0] : I \circ g = I \text{ for all } g \in G_n(\overline{K})\}.$$

Since  $K[X_n^0] = K[X_n][\Delta^{-1}]$ , we conclude that

$$K[X_n^0]^{G_n} = (K[X_n][\Delta^{-1}])^{G_n} = K[X_n]^{G_n}[\Delta^{-1}],$$

where the latter identity holds since  $\Delta$  is an invariant in  $K[X_n]^{G_n}$ . The structure of  $K[X_n^0]^{G_n}$  is deduced from Proposition 4.4 with notations from there as follows.

**Proposition 4.8.** *The ring of invariants  $K[X_n^0]^{G_n}$  of  $X_n^0$  ( $n \leq 5$ ) over a field  $K$  is*

$$\begin{cases} K[c_4, c_6][\Delta^{-1}], & \text{if } \text{char}(K) \neq 2, 3; \\ K[a_1, \Delta][\Delta^{-1}], & \text{if } \text{char}(K) = 2; \\ K[b_2, \Delta][\Delta^{-1}], & \text{if } \text{char}(K) = 3. \end{cases}$$

Proposition 4.6 yields a ring homomorphism  $I : \mathcal{M}^!(K) \rightarrow K[X_n^0]^{G_n}$  defined by  $\mathcal{F} \mapsto I_{\mathcal{F}}$ . There are, however, even more to this.

**Theorem 4.9.** *Let  $K$  be a field, the map  $I : \mathcal{M}^!(K) \rightarrow K[X_n^0]^{G_n}$  ( $n \leq 4$ ) is an isomorphism. The statement also holds to the case  $n = 5$  if  $\text{char}(K) \neq 2$ .*

*Proof.* When  $\text{char}(K) \neq 2$  or  $3$ , we know from Lemma 4.7 that on  $X_n^0$ :  $I_{\mathcal{E}_4} = c_4$ ,  $I_{\mathcal{E}_6} = -c_6$  and  $I_{\mathcal{D}} = \Delta$ . Thus the ring homomorphism  $I$  is bijective from Propositions 4.3, 4.8, formula (20) and the algebraic independence of  $\mathcal{E}_4$  and  $\mathcal{E}_6$ ,  $c_4$  and  $c_6$ . Hence  $I$  is an isomorphism.

In the case  $\text{char}(K) = 2$ ,  $I$  sends  $\mathcal{A}_2$  to  $\alpha a_1$  for some constant  $\alpha \in \mathbb{F}_2^*$ , i.e.,  $\alpha = 1$ . This comes from the fact that  $a_1$  is (up to constants) the only invariant of weight one. Moreover,  $I$  sends  $\mathcal{D}$  to  $\Delta$  by Lemma 4.7 and is thus an isomorphism from the algebraic independence of  $\mathcal{A}_2$  and  $\mathcal{D}$ ,  $a_1$  and  $\Delta$ . The independence of  $\mathcal{A}_2$  and  $\mathcal{D}$  is deduced from the independence of  $a_1$  and  $\Delta$  since  $I_{\mathcal{A}_2} = \alpha a_1$  and  $I_{\mathcal{D}} = \Delta$  on  $X_n^0$ . The case  $\text{char}(K) = 3$  is treated similarly with the identities  $I_{\mathcal{A}_3} = \beta b_2$  and  $I_{\mathcal{D}} = \Delta$  on  $X_n^0$ . Here  $\beta$  is some constant in  $\mathbb{F}_3^*$ .  $\square$

Back to our purpose, the key result in this section (which is Theorem 1.2 in Section 1) is the following:

**Theorem 4.10.** *Let  $\mathcal{F}$  be a geometric modular form of weight  $k$ , there exists a constant  $\alpha_n \in K^*$  ( $n \leq 4$ ) such that*

$$I_{\mathcal{F}}(\phi) = \alpha_n^k I_{\mathcal{F}}(\varphi_n(\phi))$$

*for any smooth genus one model  $(C_\phi, \omega_\phi)$  of degree  $n$  over a field  $K$  with the Jacobian  $(E_\phi, \omega_{\varphi_n(\phi)})$  constructed by the map  $\varphi_n$ . Moreover,  $\alpha_2 = \pm 1$ ,  $\alpha_3 = \pm 1/2$ ,  $\alpha_4 = \pm 2$ .*

From Proposition 4.8 we know that the ring of invariants of  $X_n^0$  is generated by elements of even weights except in the case of characteristic 2 in which  $1 = -1$ . Therefore, we can in any case forget the about the sign of  $\alpha_n$  and this gives a proof for Theorem 1.2.

*Proof.* We consider the map  $\varphi_n : X_n \rightarrow W$  defined in Section 3, there exists  $\alpha_n = \alpha_n(\phi) \in K^*$  depending on  $\varphi_n$  and  $\phi$  such that  $\varphi_n^*(\omega_{\varphi_n(\phi)}) = \alpha_n \omega_\phi$  for any  $\phi \in X_n^0$ . We

have for any  $\phi \in X_n^0$

$$\begin{aligned} I_{\mathcal{F}}(\varphi_n(\phi)) &= \mathcal{F}(E_\phi, \omega_{\varphi_n(\phi)}) = \mathcal{F}(C_\phi, \varphi_n^* \omega_{\varphi_n(\phi)}) \\ &= \mathcal{F}(C_\phi, \alpha_n \omega_\phi) = \alpha_n^{-k} \mathcal{F}(C_\phi, \omega_\phi) = \alpha_n^{-k} I_{\mathcal{F}}(\phi) \end{aligned}$$

and hence

$$I_{\mathcal{F}}(\phi) = \alpha_n^k I_{\mathcal{F}}(\varphi_n(\phi)). \quad (25)$$

Since  $\alpha_n$  does not depend on  $k$ , we can consider the case in which  $k = 12$  and  $\mathcal{F} = \mathcal{D}$  to get the identities  $\Delta_\phi = I_{\mathcal{D}}(\phi)$ ,  $\Delta_{\varphi_n(\phi)} = I_{\mathcal{D}}(\varphi_n(\phi))$  from Lemma 4.7 for the corresponding geometric modular form  $\mathcal{D} = (\mathcal{E}_4^3 - \mathcal{E}_6^2)/1728$ . This deduces

$$\Delta_\phi = \alpha_n^{12} \Delta_{\varphi_n(\phi)}.$$

As proved in Proposition 3.2,  $\Delta_\phi = c \Delta_{\varphi_n(\phi)}$  for some constant  $c$  when  $n \leq 4$ . More precisely,

$$\begin{cases} c = \pm 1, & \text{If } n = 2; \\ c = \pm 2^{-12}, & \text{If } n = 3; \\ c = \pm 2^{12}, & \text{If } n = 4. \end{cases}$$

We now need to compute  $\alpha_n$  from  $\alpha_n^{12} = c$ . Look again at (25) to the case  $\mathcal{F} = \mathcal{E}_4$  and  $\mathcal{E}_6$ , we have  $c_4(\phi) = \alpha_n^4 c_4(\varphi_n(\phi))$  and  $c_6(\phi) = \alpha_n^6 c_6(\varphi_n(\phi))$  for any  $\phi \in X_n^0$ . Consider a model  $\phi$  with integer coefficients, we conclude from Lemma 2.2 that  $\alpha_n^4, \alpha_n^6 \in \mathbb{Q}$  and thus  $\alpha_n^2 \in \mathbb{Q}$ . This enables us to exclude the minus sign of the constant  $c$  above and deduce the discussion at the end of Section 3. We have in addition

$$\alpha_2^2 = 1, \alpha_3^2 = 1/4 \text{ and } \alpha_4^2 = 4$$

or

$$\alpha_2 = \pm 1, \alpha_3 = \pm 1/2 \text{ and } \alpha_4 = \pm 2.$$

□

Observe that the formulae of  $c_4, c_6$  obtained from Theorem 1.2 for  $k = 4, 6$  respectively in cases  $n = 2, 3, 4$  are the same with the normalized ones given by Fisher as in (13), (14) and (15).

## 5. DETERMINANTAL REPRESENTATIONS OF WEIERSTRASS CUBICS

We will in this section study discriminants of smooth curves in Weierstrass form and provide a proof to Theorem 1.5. Consider a smooth curve  $C_\phi$  given by

$$\phi(x, y, z) = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3, \quad (26)$$

where  $g_2$  and  $g_3$  are elements in a field  $K$ . We want to find the  $3 \times 3$  square matrices  $L, M, N$  such that

$$\det(xL + yM + zN) = \phi(x, y, z).$$

We obtain from [21, Section 2] the following determinantal representations of  $\phi$

$$\begin{pmatrix} 2x + tz & y + dz & (3t^2 - g_2)z \\ 0 & x - tz & y - dz \\ z & 0 & -2x - tz \end{pmatrix}, \quad (27)$$

with  $t, d \in \overline{K}$  be such that  $d^2 = 4t^3 - g_2t - g_3$ . It can be checked that the determinant of (27) is equal to  $\phi$ .

Now we move to the theory of theta functions to study the case when  $K = \mathbb{C}$ . The following discussion bases on Wang and Guo [22]. In this case, there exists a unique lattice  $\Lambda$  coming from the Weierstrass parametrization such that  $C_\phi(\mathbb{C}) \cong \mathbb{C}/\Lambda$ . Here  $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$  for some  $\omega_1, \omega_2 \in \mathbb{C}$  with  $\tau = \omega_2/\omega_1 \in \mathbb{H}$ . The two coefficients  $g_2$  and  $g_3$  of the curve given by  $\phi$  can be determined by (see [22, p. 509])

$$g_2 = \frac{2}{3} \left( \frac{\pi}{\omega_1} \right)^4 (a^8 + b^8 + c^8),$$

$$g_3 = \frac{4}{27} \left( \frac{\pi}{\omega_1} \right)^6 (a^4 + b^4)(b^4 + c^4)(c^4 - a^4),$$

where  $a = \theta_2(0, \tau) = e^{\frac{\pi i \tau}{4}} \theta(\frac{1}{2}\tau, \tau)$ ,  $b = \theta_3(0, \tau) = \theta(0, \tau)$  and  $c = \theta_4(0, \tau) = \theta(\frac{1}{2}, \tau)$  with the even Jacobi theta functions:

$$\theta(z, \tau) = \theta_3(z, \tau) := \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z),$$

$$\theta_2(z, \tau) = \exp(\pi i \tau / 4 + \pi i z) \theta(z + \tau/2, \tau),$$

$$\theta_4(z, \tau) = \theta(z + \tau/2, \tau).$$

The above  $a, b, c$  are called even theta constants.

Since  $(t, d)$  is a point on the affine curve associated to  $C_\phi$  defined by  $\{z \neq 0\}$ , it is determined by theta constants via Weierstrass  $\mathcal{P}$ -function and so are all the coefficients in the linear matrix (27). To be precise, we consider the Weierstrass  $\mathcal{P}$ -function associated to the lattice  $\Lambda$  defined for all  $s \notin \Lambda$  as

$$\mathcal{P}(s) = \mathcal{P}(s; \omega_1, \omega_2) := \frac{1}{s^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \left( \frac{1}{(s + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right).$$

As in [22, p. 469], it satisfies the differential equation

$$\mathcal{P}'(s)^2 = 4\mathcal{P}(s)^3 - g_2\mathcal{P}(s) - g_3.$$

We can parametrize the point  $(t, d)$  on the curve as  $t = \mathcal{P}(s)$  and  $d = \mathcal{P}'(s)$  for some  $s \notin \Lambda$ . It is known that the discriminant of the cubic (26) is given by the formula (see Remark 1.3)

$$\Delta_\phi = 2^{12}(g_2^3 - 27g_3^2) = 2^{16} \left( \frac{\pi}{\omega_1} \right)^{12} (abc)^8. \quad (28)$$

We will give another proof for the formula (28) using resultant and the determinantal representation (27). From [15, p. 434], the discriminant of a homogeneous cubic polynomial  $\phi(x, y, z)$  can be computed by resultant defined there as

$$\Delta_\phi = -\text{Res}(\phi_x, \phi_y, \phi_z)/27. \quad (29)$$

The reader can have a look at [15, Chapter 13] for a general discussion about resultants. We choose the minus sign here so that the sign of the discriminant is compatible to other sections of the paper. To simplify the computation, we choose a special value for the Weierstrass function  $\mathcal{P}(s)$ , namely, we choose the 2-torsion point  $s = \omega_2/2$ . In this case  $\mathcal{P}(\omega_2/2) = -\frac{\pi^2}{3\omega_1^2}(a^4 + b^4)$  and  $\mathcal{P}'(\omega_2/2) = 0$  by [22, p. 470, 509]. Then  $d = 0$  and  $t = -\frac{\pi^2}{3\omega_1^2}(a^4 + b^4)$ . Besides, using the Jacobi's identity  $a^4 + c^4 = b^4$  (see [22, p. 504]), the matrix (27) above can be written in the form

$$\begin{pmatrix} 2x - \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z & y & -(\frac{\pi}{\omega_1})^4 c^8 z \\ 0 & x + \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z & y \\ z & 0 & -2x + \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z \end{pmatrix}. \quad (30)$$

We have thus proved Theorem 1.5. From the representation  $\phi = \det(U)$ , where  $U$  is given by (30), we get that

$$\phi_x = -12x^2 + \frac{2}{3} \left( \frac{\pi}{\omega_1} \right)^4 (a^8 + b^8 + c^8)z^2,$$

$$\phi_y = 2yz, \text{ and}$$

$$\phi_z = y^2 + \frac{4}{3} \left( \frac{\pi}{\omega_1} \right)^4 (a^8 + b^8 + c^8)xz + \frac{4}{9} \left( \frac{\pi}{\omega_1} \right)^6 (a^4 + b^4)(b^4 + c^4)(c^4 - a^4)z^2.$$

The discriminant  $\Delta_\phi$  of the cubic  $\phi$  is then obtained via (29)

$$\Delta_\phi = 2^{16} \left( \frac{\pi}{\omega_1} \right)^{12} (abc)^8.$$

In fact, we can directly use (29) to the curve (26). But this approach of determinantal representations might be applied to more general cases. We will explain in more detail in Section 7.

## 6. DETERMINANTAL REPRESENTATIONS OF COMPLEX PLANE CURVES

In Section 5, we have already seen that one can compute the discriminant of smooth curves over  $\mathbb{C}$  in Weierstrass form by using determinantal representations. Our goal is

to generalize to plane curves of arbitrary degrees based on Theorem 5.1 in [3]. In other words, we will in this section prove Theorem 6.3. Let us first introduce some notations.

Let  $X$  be a compact Riemann surface, let  $\mathcal{L}$  is a line bundle of half differentials on  $X$  (a theta characteristic), i.e.,  $\mathcal{L}^{\otimes 2}$  is the canonical bundle  $\omega_X$  on  $X$  and let  $\chi$  be a flat line bundle over  $X$  such that  $h^0(\chi \otimes \mathcal{L}) = 0$ . We associate to  $\chi$  the Cauchy kernel  $K(\chi; \cdot, \cdot)$  as defined in Section 2 of [3]. Let  $\lambda_1, \lambda_2$  be two scalar meromorphic functions on  $X$ , which generate the whole field of meromorphic functions. Assume that all poles of  $\lambda_1, \lambda_2$  are simple and labeled as  $P_1, \dots, P_d \in X$ . We write the Laurent expansion of  $\lambda_k$  at  $P_i$  ( $1 \leq i \leq d, k = 1, 2$ ) with some fixed local coordinate  $t_i = t_i(P)$  centered at  $P = P_i$

$$\lambda_k(P) = -\frac{c_{ik}}{t_i} - d_{ik} + O(|t_i|).$$

Then we define the  $d \times d$  matrices  $L, M, N$  by

$$L = \text{diag}_{1 \leq i \leq d}(c_{i2}), \quad M = \text{diag}_{1 \leq i \leq d}(-c_{i1}), \quad N = (n_{ij})_{i,j},$$

where

$$n_{ij} = \begin{cases} d_{i1}c_{i2} - d_{i2}c_{i1}, & i = j; \\ (c_{i1}c_{j2} - c_{j1}c_{i2}) \frac{K(\chi; P_i, P_j)}{dt_j(P_j)}, & i \neq j. \end{cases}$$

The result mentioned in [3] is the following

**Proposition 6.1.** *The map  $\pi_0 : X \rightarrow \mathbb{C}^2$  given by  $\pi_0(P) = (\lambda_1(P), \lambda_2(P))$  maps  $X \setminus \{P_1, \dots, P_d\}$  onto the affine part  $C^0$  of an algebraic curve  $C \subset \mathbb{P}^2$  and extends to a proper birational map  $\pi : X \rightarrow C$  of  $X$  in  $\mathbb{P}^2$ . The defining irreducible homogeneous polynomial  $\phi(x, y, z)$  of  $C$  is such that (up to multiplying by some constant)*

$$\phi(x, y, z) = \det(xL + yM + zN).$$

Here the affine part  $C^0$  of  $C$  is defined by  $\{z \neq 0\}$ .

The authors in [3] prove a more general version of the above proposition where they consider  $\chi$  to be any flat vector bundle. We restrict here to the case of line bundle since it is enough for our purpose.

Suppose in this case that  $\chi$  is defined by a unitary representation of the fundamental group of  $X$  given by

$$\chi(\alpha_i) = \exp(-2\pi i a_i) \text{ and } \chi(\beta_i) = \exp(2\pi i b_i), \quad i = 1, \dots, g,$$

where  $a_i, b_i \in \mathbb{R}$ ,  $g$  is the genus of  $X$  and  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$  form a symplectic basis of  $H_1(X, \mathbb{Z})$ . Let  $(\eta_1, \dots, \eta_g)$  be a basis of holomorphic 1-forms on  $X$ , we form the period matrix with respect to these bases which is the  $g \times 2g$ -matrix  $(\Omega_1 \mid \Omega_2)$  whose entries

are

$$(\Omega_1)_{ij} = \int_{\alpha_j} \eta_i \text{ and } (\Omega_2)_{ij} = \int_{\beta_j} \eta_i, \text{ for } i, j = 1, \dots, g.$$

We choose the canonical basis  $(\eta_1, \dots, \eta_g)$  of holomorphic 1-forms in the sense that  $\int_{\alpha_i} \eta_j = \delta_{ij}$ , then the corresponding period matrix will be of the form  $(I_g \mid \Omega)$ . The matrix  $\Omega$  lies in the Siegel upper half space  $\mathbb{H}^g$  and it is called the Riemann period matrix of  $X$  with respect to the homology basis  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ . We fix such a symplectic homology basis and the resulting period matrix  $\Omega$ . Let  $J(X) = \mathbb{C}^g / (\mathbb{Z}^g + \Omega\mathbb{Z}^g)$  be the Jacobian of  $X$  and  $\varphi : X \rightarrow J(X)$  be the Abel-Jacobi map with any fixed base point. Then we have an explicit formula for the Cauchy kernel (see [3, Theorem 4.1]).

$$K(\chi; P, Q) = \frac{\theta[\delta](\varphi(Q) - \varphi(P))}{\theta[\delta](0)E(Q, P)},$$

where  $\theta[\delta]$  is the associated theta function with characteristic  $\delta = b + \Omega a = \varphi(\chi)$  ( $a = (a_j)_j$  and  $b = (b_j)_j$ ) and  $E(\cdot, \cdot)$  is the prime form on  $X \times X$ . Recall from [11, Chapter II] that the prime form  $E$  is a bi-half-differential with simple poles along the diagonal of  $X \times X$ .

Here the theta characteristic  $\mathcal{L}$  is chosen such that  $\varphi(\mathcal{L}) = -\mathcal{K}$ , where  $\mathcal{K}$  is the vector of Riemann constants. Note that a consequence of the Riemann singularity theorem is that  $\theta(b + \Omega a) \neq 0$  if and only if  $h^0(\chi \otimes \mathcal{L}) = 0$ . Hence  $\theta[\delta](0) \neq 0$  and the formula above makes sense.

From this proposition, one can provide explicitly determinantal representations for complex plane curves using theta functions and the Abel-Jacobi map. The reader can have a look at [17, Section 4], [20, Theorem 6] or [6, Theorem 2.2] for reference. Note that results in the reference above only apply to the family of hyperbolic curves with a normalization, but it can be written in the following general form.

**Theorem 6.2.** *Let  $C_\phi \subset \mathbb{P}^2$  be a non-rational irreducible complex plane curve defined by  $\phi = 0$ , where  $\phi(x, y, z)$  is an irreducible homogeneous polynomial of degree  $d$ . Suppose the  $d$  intersection points of  $C_\phi$  with the line  $\{y = 0\}$  are distinct non-singular points  $P_1, \dots, P_d$  with coordinates  $P_i = (1, 0, \beta_i)$ ,  $\beta_i \neq 0$ . Then*

$$\phi(x, y, z) = \lambda \det(xM + yN + zI),$$

where  $\lambda = \phi(0, 0, 1)$ ,  $M = \text{diag}(-\beta_1, \dots, -\beta_d)$  and  $N = (n_{ij})_{i,j}$  with

$$n_{ii} = -\beta_i \frac{\phi_y(1, 0, \beta_i)}{\phi_x(1, 0, \beta_i)}$$

and for  $i \neq j$

$$n_{ij} = \frac{\beta_i - \beta_j}{\theta[\delta](0)} \cdot \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{E(P_j, P_i)} \cdot \frac{1}{\sqrt{d(-y/x)(P_i)} \sqrt{d(-y/x)(P_j)}}.$$



Here  $\delta$  is an even theta characteristic such that  $\theta[\delta](0) \neq 0$ ,  $\varphi : X \rightarrow J(X)$  is the Abel-Jacobi map from the desingularizing Riemann surface  $X$  of  $C_\phi$  to its Jacobian and  $E(.,.)$  is the prime form on  $X \times X$ .

We want to generalize Theorem 6.2 such that we replace the line  $\{y = 0\}$  by a general line passing through distinct points of  $C_\phi$ . Let  $l$  be a line defined by  $\alpha x + \beta y + \gamma z = 0$  so that its affine part  $l^0$  defined by  $\alpha x + \beta y + \gamma = 0$  intersects the affine part  $C_\phi^0$  of  $C_\phi$  at  $d$  distinct non-singular points  $P_i^0$ ,  $i = 1, \dots, d$ . Since  $\alpha$  and  $\beta$  can not be both zero, we can suppose w.l.o.g that  $\beta \neq 0$  (the case  $\alpha \neq 0$  can be treated similarly). In this case, we can suppose further that  $\beta = -1$ . Therefore, the line  $l$  can be rewritten as  $y = \alpha x + \gamma z$ . Assume that the intersections points  $P_i^0$  of  $l^0$  and  $C_\phi^0$  have non-zero  $x$ -coordinates so that we can write  $P_i^0 = (1/\beta_i, \alpha/\beta_i + \gamma)$  with  $\beta_i \neq \beta_j$  if  $i \neq j$ . Thus the intersection points of  $l$  and  $C_\phi$  are  $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$ . We now prove the following

**Theorem 6.3.** *Let  $C_\phi \subset \mathbb{P}^2$  be a non-rational irreducible complex plane curve defined by  $\phi = 0$ , where  $\phi(x, y, z)$  is an irreducible homogeneous polynomial of degree  $d$ . Suppose the  $d$  intersection points of  $C_\phi$  with the line  $\{y = \alpha x + \gamma z\}$  are distinct non-singular points  $P_1, \dots, P_d$  with coordinates  $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$ ,  $\beta_i \neq 0$ . Then up to multiplying by some constant*

$$\phi(x, y, z) = \det((M - \alpha N)x + Ny + (I - \gamma N)z),$$

where  $M = \text{diag}(-\beta_1, \dots, -\beta_d)$  and  $N = (n_{ij})_{i,j}$  with

$$n_{ii} = -\frac{\beta_i \phi_y(P_i)}{(\phi_x + \alpha \phi_y)(P_i)}$$

and for  $i \neq j$

$$n_{ij} = \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{\theta[\delta](0)E(P_j, P_i)} \frac{\beta_i - \beta_j}{\sqrt{\beta_i(\alpha dx - dy)(P_i)} \sqrt{\beta_j(\alpha dx - dy)(P_j)}}.$$

Here  $\delta$  is an even theta characteristic such that  $\theta[\delta](0) \neq 0$ ,  $\varphi : X \rightarrow J(X)$  is the Abel-Jacobi map from the desingularizing Riemann surface  $X$  of  $C_\phi$  to its Jacobian and  $E(.,.)$  is the prime form on  $X \times X$ .

*Proof.* Apply Proposition 6.1 with the pair of meromorphic functions on the desingularizing Riemann surface  $X$  of  $C_\phi$ :

$$\lambda_1 = \frac{1}{y - \alpha x - \gamma}, \quad \lambda_2 = \frac{x}{y - \alpha x - \gamma}$$

and  $t = \frac{\alpha x - y + \gamma}{x}$  be the local coordinates at the poles  $P_i$  (zeros of  $\alpha x - y + \gamma$ ). The next step is to write down Laurent expansions of  $\lambda_1, \lambda_2$  at  $P_i$ . We have

$$\lambda_2 = -1/t \Rightarrow c_{i2} = 1, \quad d_{i2} = 0 \quad \forall i.$$

At  $P_i$  we have  $\lambda_1 = -\frac{1}{t}(\frac{1}{x})$ . Since

$$\frac{1}{x} = \beta_i + \frac{d(\frac{1}{x})}{d(\frac{\alpha x - y + \gamma}{x})}(P_i)t + O(|t|^2) = \beta_i + \beta_i \frac{dx}{d(y - \alpha x)}(P_i)t + O(|t|^2),$$

we deduce that

$$c_{i1} = \beta_i, \quad d_{i1} = \beta_i \frac{dx}{d(y - \alpha x)}(P_i).$$

We then obtain from Proposition 6.1 that (up to some constant)

$$\phi = \det((M - \alpha N)x + Ny + (I - \gamma N)z)$$

where  $M = \text{diag}(-\beta_1, \dots, -\beta_d)$  and  $N = (n_{ij})$  with

$$n_{ii} = \beta_i \frac{dx}{d(y - \alpha x)}(P_i)$$

and for  $i \neq j$

$$n_{ij} = \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{\theta[\delta](0)E(P_j, P_i)} \frac{\beta_i - \beta_j}{d(\frac{\alpha x - y + \gamma}{x})(P_j)}.$$

Here  $\delta$  is an even theta characteristic with  $\theta[\delta](0) \neq 0$ . Note that the affine part  $C_\phi^0$  of  $C_\phi$  is defined by

$$\{(\lambda_1(P), \lambda_2(P)) \mid P \in X \setminus \{P_1, \dots, P_d\}\}.$$

Furthermore, if we replace  $N$  by the matrix  $N'$  which has the same diagonal elements with  $N$  but different off-diagonal elements

$$n'_{ij} = \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{\theta[\delta](0)E(P_j, P_i)} \frac{\beta_i - \beta_j}{\sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_i)} \sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_j)}},$$

then the determinantal representation does not change. Indeed, let

$$U = (M - \alpha N)x + Ny + (I - \gamma N)z$$

and

$$U' = (M - \alpha N')x + N'y + (I - \gamma N')z,$$

if we multiply the  $i^{\text{th}}$ -column of  $U$  and the  $i^{\text{th}}$ -row of  $U'$  (for  $i = 1, \dots, d$ ) with the term  $\sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_i)}$  then both of them will become the same matrix  $U^*$ . Consequently,

$$\det(U) = \det(U') = \frac{\det(U^*)}{\prod_{i=1}^d \sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_i)}}.$$

Observe that  $d(\frac{\alpha x - y + \gamma}{x})(P_i) = \beta_i(\alpha dx - dy)(P_i)$  and

$$\frac{dx}{d(y - \alpha x)}(P_i) = -\frac{\phi_y(P_i)}{(\phi_x + \alpha \phi_y)(P_i)}$$

by implicit function theorem with the fact that  $(\phi_x + \alpha\phi_y)(P_i) \neq 0$ . Indeed, since the polynomial  $f(x) := \phi(x, \alpha x + \gamma, 1)$  has distinct roots  $(1/\beta_i)$  we conclude that  $f'(1/\beta_i) \neq 0$  and hence  $(\phi_x + \alpha\phi_y)(P_i) \neq 0$ . We have thus proved Theorem 6.3.  $\square$

Theorem 6.2 is then established by reducing to the case  $\alpha = \gamma = 0$ . We will apply Theorem 6.3 in the next section to get a formula for the discriminant of plane cubic curves.

**Remark 6.4.** *We can also reformulate the analogous statement to the Theorem 6.3 if the line  $y = \alpha x + \gamma z$  is replaced by  $x = \alpha y + \gamma z$ .*

## 7. DISCRIMINANT OF PLANE CUBIC CURVES

We now study the main object of interest in which we consider a smooth plane curve  $C_\phi$  over  $\mathbb{C}$  defined by the cubic form  $\phi = 0$ . The affine part of the curve  $C_\phi$  is parametrized as

$$\{(x, y, 1) = (R_1(\mathcal{P}(s), \mathcal{P}'(s)), R_2(\mathcal{P}(s), \mathcal{P}'(s)), 1)\},$$

where  $\mathcal{P}(s; \omega_1, \omega_2)$  is the Weierstrass  $\mathcal{P}$ -function associated to some  $\omega_1, \omega_2 \in \mathbb{C}$  satisfying  $\text{Im}(\omega_2/\omega_1) > 0$ .

In this section, we use the standard notation  $\tau$  of genus one case instead of  $\Omega$  for the period matrix. Moreover, we use the general Jacobian  $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$  in place of the normalized one  $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  for  $\tau = \omega_2/\omega_1$  in order to use the properties of the function  $\mathcal{P}$ . By this change, an extra factor  $1/\omega_1$  appears in the below elements  $n_{ij}$  ( $i \neq j$ ) in comparing with Theorem 6.3. This idea was mentioned in [6, Theorem 2.4]. In addition, we use the notation  $\theta_\delta$  ( $\delta = 1, 2, 3, 4$ ) for theta functions as in Section 5 instead of  $\theta[\delta]$ .

The prime form  $E(P, Q)$  in genus one case is better understood so that we obtain a consequence of Theorem 6.3 as follows.

**Corollary 7.1.** *Let  $C_\phi \subset \mathbb{P}^2$  be a smooth plane cubic curve defined by  $\phi = 0$ , where  $\phi(x, y, z)$  is a non-singular homogeneous cubic polynomial. Suppose the line  $y = \alpha x + \gamma z$  intersects  $C_\phi$  at 3 distinct points  $P_1, P_2, P_3$  with coordinates  $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$ ,  $\beta_i \neq 0$ . Then up to multiplying by some constant*

$$\phi(x, y, z) = \det((M - \alpha N)x + Ny + (I - \gamma N)z),$$

where  $M = \text{diag}(-\beta_1, -\beta_2, -\beta_3)$  and  $N = (n_{ij})_{i,j}$  with

$$n_{ii} = -\frac{\beta_i \phi_y(P_i)}{(\phi_x + \alpha \phi_y)(P_i)}$$

and for  $i \neq j$

$$n_{ij} = \frac{\theta'_1(0)\theta_\delta((Q_j - Q_i)/\omega_1)}{\omega_1\theta_\delta(0)\theta_1((Q_j - Q_i)/\omega_1)} \frac{\beta_i - \beta_j}{\sqrt{\beta_i(\alpha R'_1 - R'_2)(Q_i)}\sqrt{\beta_j(\alpha R'_1 - R'_2)(Q_j)}}$$

Here  $\delta$  is any even theta characteristic, i.e.,  $\delta = 2, 3$  or  $4$  and  $Q_i = \varphi(P_i)$ . Note that we also have an analogous statement of this corollary by Remark 6.4.

The field of meromorphic functions on a general genus one curve is generated by  $\mathcal{P}, \mathcal{P}'$  associated to some periods  $\omega_1, \omega_2$ . Thus  $R_1$  and  $R_2$  are rational functions on  $\mathcal{P}, \mathcal{P}'$ . In general,  $R_1$  and  $R_2$  have complicated expressions. But we have better interpretations in the case of plane cubic curves. In this case,  $C_\phi$  always has a flex point and hence can be transformed to a Weierstrass equation after a linear coordinate change (see [8, Section 4.4]). Thus, we are able to present rational functions  $R_1, R_2$  as:

$$\begin{aligned} R_1(s) &= \lambda_{11}\mathcal{P}(s) + \lambda_{12}\mathcal{P}'(s) + \lambda_{13}, \\ R_2(s) &= \lambda_{21}\mathcal{P}(s) + \lambda_{22}\mathcal{P}'(s) + \lambda_{23}. \end{aligned} \tag{31}$$

The constants  $\lambda_{ij} \in \mathbb{C}$  satisfy  $\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}$  and depend on the coefficients of  $\phi$ . Here we fix any flex point and the corresponding periods  $\omega_1, \omega_2$  coming from the Weierstrass parametrization of the Weierstrass equation.

To shorten the determinantal representation, we should look at 2-torsion points to simplify  $\theta$  and  $\mathcal{P}$ . More precisely, we consider the line  $l$  which intersects  $C_\phi$  at the points  $P_i$  such that the corresponding points  $Q_i$  on the torus  $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$  are  $\omega_1/2, (\omega_1 + \omega_2)/2$  and  $\omega_2/2$  respectively. Suppose that the  $x$ -coordinates of  $P_i$  are all non-zero. We will treat the case  $l$  to have the form  $y = \alpha x + \gamma z$  and then make use of Corollary 7.1. The other case can be treated similarly using Remark 6.4. The choice of 2-torsion points gives us the convenience to work with some computations below. Let  $a = \theta_2(0, \tau), b = \theta_3(0, \tau), c = \theta_4(0, \tau)$ , where  $\tau = \omega_2/\omega_1$ , we will prove the following

**Proposition 7.2.** *Let  $C_\phi \subset \mathbb{P}^2$  be a smooth plane curve defined by  $\phi = 0$ , where  $\phi(x, y, z)$  is a non-singular homogeneous cubic polynomial. Suppose the line  $y = \alpha x + \gamma z$  intersects  $C_\phi$  at 3 distinct points  $P_1, P_2, P_3$  with coordinates  $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$ ,  $\beta_i \neq 0$  so that the corresponding points  $Q_i = \varphi(P_i)$  of  $P_i$  on the torus  $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$  are  $\omega_1/2, (\omega_1 + \omega_2)/2$  and  $\omega_2/2$  respectively. Denote by  $k = \alpha\lambda_{12} - \lambda_{22}$ , then we have the following expressions (up to some constant) for the discriminant  $\Delta_\phi$  of  $\phi$*

(1)

$$\Delta_\phi = \frac{\lambda_{11}^6 \omega_1^{24}}{2^8 k^{12} \pi^{24} (abc)^{16}} (\beta_1 - \beta_2)^6 (\beta_1 - \beta_3)^6 (\beta_2 - \beta_3)^6,$$

(2)

$$\Delta_\phi = 16 \left( \frac{\lambda_{11}^2 \pi \beta_1 \beta_2 \beta_3}{2k\omega_1} \right)^{12} (abc)^8.$$

*Proof.* By [22, p. 470, 509], we have  $\mathcal{P}'(Q_i) = 0$  for all  $i$  and

$$\mathcal{P}(Q_1) = \frac{\pi^2}{3\omega_1^2}(b^4 + c^4), \quad \mathcal{P}(Q_2) = \frac{\pi^2}{3\omega_1^2}(a^4 - c^4), \quad \mathcal{P}(Q_3) = -\frac{\pi^2}{3\omega_1^2}(a^4 + b^4).$$

Besides,  $\mathcal{P}''(s) = 6(\mathcal{P}(s))^2 - g_2/2$  with  $g_2 = \frac{2}{3}(\frac{\pi}{\omega_1})^4(a^8 + b^8 + c^8)$  as in [22, p. 469]. Thus

$$\mathcal{P}''(Q_1) = \frac{2\pi^4 b^4 c^4}{\omega_1^4}, \quad \mathcal{P}''(Q_2) = -\frac{2\pi^4 a^4 c^4}{\omega_1^4}, \quad \mathcal{P}''(Q_3) = \frac{2\pi^4 a^4 b^4}{\omega_1^4}.$$

We also have for each  $i$

$$-\frac{\phi_y(P_i)}{(\phi_x + \alpha\phi_y)(P_i)} = \frac{dx}{d(y - \alpha x)}(P_i) = \frac{R'_1}{(R'_2 - \alpha R'_1)}(Q_i) = \frac{\lambda_{12}}{\lambda_{22} - \alpha\lambda_{12}}.$$

Choosing  $\delta = 3$ , we now simplify the matrix  $N$  in Corollary 7.1. Let  $k_1 = -\lambda_{12}/k$  and note that  $\theta'_1(0) = \pi abc$  as in [22, p. 507], we have  $n_{ii} = k_1\beta_i$  and  $n_{13} = n_{31} = 0$  as  $\theta((1 + \tau)/2) = 0$ . Moreover,

$$n_{12}^2 = n_{21}^2 = \frac{\pi^2(abc)^2\theta^2\left(\frac{\tau}{2}\right)(\beta_1 - \beta_2)^2}{\omega_1^2 k^2 b^2 \theta_1^2\left(\frac{\tau}{2}\right)\beta_1\beta_2\mathcal{P}''(Q_1)\mathcal{P}''(Q_2)} = \frac{\omega_1^6(\beta_1 - \beta_2)^2}{4k^2\pi^6\beta_1\beta_2b^4c^8},$$

$$n_{23}^2 = n_{32}^2 = \frac{\pi^2(abc)^2\theta^2\left(\frac{1}{2}\right)(\beta_2 - \beta_3)^2}{\omega_1^2 k^2 b^2 \theta_1^2\left(\frac{1}{2}\right)\beta_2\beta_3\mathcal{P}''(Q_2)\mathcal{P}''(Q_3)} = -\frac{\omega_1^6(\beta_2 - \beta_3)^2}{4k^2\pi^6\beta_2\beta_3a^8b^4}.$$

Here we use the fact that (see [22, p. 502])

$$\theta\left(\frac{\tau}{2}\right) = q^{-\frac{1}{8}}a, \theta_1\left(\frac{\tau}{2}\right) = iq^{-\frac{1}{8}}c, \theta\left(\frac{1}{2}\right) = c, \theta_1\left(\frac{1}{2}\right) = a$$

with  $q = e^{2\pi i\tau}$ . We have  $1/\beta_i = \lambda_{11}\mathcal{P}(Q_i) + \lambda_{13}$  from (31) and the fact  $R_1(Q_i) = 1/\beta_i$ . Therefore,

$$\begin{aligned} \frac{\beta_1 - \beta_2}{\beta_1\beta_2} &= \lambda_{11}(\mathcal{P}(Q_2) - \mathcal{P}(Q_1)) = -\lambda_{11}\frac{\pi^2 c^4}{\omega_1^2}, \\ \frac{\beta_1 - \beta_3}{\beta_1\beta_3} &= \lambda_{11}(\mathcal{P}(Q_3) - \mathcal{P}(Q_1)) = -\lambda_{11}\frac{\pi^2 b^4}{\omega_1^2}, \\ \frac{\beta_2 - \beta_3}{\beta_2\beta_3} &= \lambda_{11}(\mathcal{P}(Q_3) - \mathcal{P}(Q_2)) = -\lambda_{11}\frac{\pi^2 a^4}{\omega_1^2}. \end{aligned} \tag{32}$$

It can be seen from (32) that  $\lambda_{11} \neq 0$ . Similarly we have  $\lambda_{21} = \alpha\lambda_{11}$  from the identities  $R_2(Q_i) = \alpha/\beta_i + \gamma$ . Breaking out the determinant, one get the following expression for  $\phi$  (up to some constant  $\lambda$ )

$$\begin{aligned} & -\beta_1\beta_2\beta_3x^3 + 3\beta_1\beta_2\beta_3k_1x^2y + (\beta_3n_{12}^2 + \beta_1n_{23}^2 - 3\beta_1\beta_2\beta_3k_1^2)xy^2 + \\ & k_1(\beta_1\beta_2\beta_3k_1^2 - \beta_3n_{12}^2 - \beta_1n_{23}^2)y^3 + (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)x^2z - 2k_1(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)xyz \\ & + (k_1^2(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3) - n_{12}^2 - n_{23}^2)y^2z - (\beta_1 + \beta_2 + \beta_3)xz^2 + k_1(\beta_1 + \beta_2 + \beta_3)yz^2 + z^3. \end{aligned} \tag{33}$$

Consequently,  $\text{Res}(\phi_x/\lambda, \phi_y/\lambda, \phi_z/\lambda) =$

$$(-432)(\beta_2 - \beta_3)^2(\beta_1 - \beta_2)^2n_{23}^4n_{12}^4(\beta_1 - \beta_3)^6(\beta_2n_{12}^2 - \beta_3n_{12}^2 - \beta_1n_{23}^2 + \beta_2n_{23}^2)^2.$$

The term  $\beta_2 n_{12}^2 - \beta_3 n_{12}^2 - \beta_1 n_{23}^2 + \beta_2 n_{23}^2$  is equal to

$$\begin{aligned} & \frac{(\beta_1 - \beta_2)(\beta_2 - \beta_3)\omega_1^6}{4k^2\pi^6b^4} \left( \frac{\beta_1 - \beta_2}{c^8\beta_1\beta_2} + \frac{\beta_2 - \beta_3}{a^8\beta_2\beta_3} \right) \\ &= -\frac{\lambda_{11}\omega_1^4(\beta_1 - \beta_2)(\beta_2 - \beta_3)}{4k^2\pi^4a^4c^4}, \end{aligned}$$

where the later equality comes from (32). Furthermore,

$$(n_{12}n_{23})^4 = \frac{\omega_1^{24}(\beta_1 - \beta_2)^4(\beta_2 - \beta_3)^4}{2^8k^8\pi^{24}(abc)^{16}(\beta_1\beta_2)^2(\beta_2\beta_3)^2} = \frac{\lambda_{11}^4\omega_1^{16}(\beta_1 - \beta_2)^2(\beta_2 - \beta_3)^2}{2^8k^8\pi^{16}a^8b^{16}c^8}.$$

The later equality again comes from (32). Hence

$$\Delta_\phi = -\frac{1}{27}\text{Res}(\phi_x, \phi_y, \phi_z) = \frac{\lambda^{12}\lambda_{11}^6\omega_1^{24}}{2^8k^{12}\pi^{24}(abc)^{16}}(\beta_1 - \beta_2)^6(\beta_1 - \beta_3)^6(\beta_2 - \beta_3)^6.$$

We also obtain an alternative form of the discriminant by using (32):

$$\Delta_\phi = 16 \left( \frac{\lambda\lambda_{11}^2\pi\beta_1\beta_2\beta_3}{2k\omega_1} \right)^{12} (abc)^8. \quad (34)$$

This completes the proof of Proposition 7.2.  $\square$

We now simplify the formula (34) by looking at the relationships between  $\lambda, \lambda_{11}, k$  and  $\beta_1\beta_2\beta_3$ . It can be seen from (33) that  $\lambda\beta_1\beta_2\beta_3 = -\phi(1, 0, 0)$ . The transformation (31) means that if we write

$$x = \lambda_{11}X + \lambda_{12}Y + \lambda_{13},$$

$$y = \lambda_{21}X + \lambda_{22}Y + \lambda_{23}$$

then the affine curve  $\phi(x, y, 1) = 0$  will be transformed to a Weierstrass form  $-Y^2 + 4X^3 - g_2X - g_3 = 0$ . In addition, the inverse transformation

$$X = l_{11}x + l_{12}y + l_{13},$$

$$Y = l_{21}x + l_{22}y + l_{23}$$

would transform the Weierstrass equation  $-Y^2 + 4X^3 - g_2X - g_3 = 0$  to:

$$\begin{aligned} & 4l_{11}^3x^3 + 12l_{11}^2l_{12}x^2y + 12l_{11}l_{12}^2xy^2 + 4l_{12}^3y^3 + (12l_{11}^2l_{13} - l_{21}^2)x^2 + \\ & (24l_{11}l_{12}l_{13} - 2l_{21}l_{22})xy + (12l_{12}^2l_{13} - l_{22}^2)y^2 + (12l_{11}l_{13}^2 - 2l_{21}l_{23} - l_{11}g_2)x + \\ & (12l_{12}l_{13}^2 - 2l_{22}l_{23} - l_{12}g_2)y + 4l_{13}^3 - l_{23}^2 - l_{13}g_2 - g_3. \end{aligned} \quad (35)$$

One can check that  $l_{11} = \lambda_{22}/D, l_{12} = -\lambda_{12}/D, l_{21} = -\lambda_{21}/D$  and  $l_{22} = \lambda_{11}/D$  with  $D = \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}$ . Compare the coefficients of  $x^3$  and  $x^2y$  in (33) and (35), we have

$$\begin{cases} 4l_{11}^3 = -\lambda\beta_1\beta_2\beta_3, \\ 12l_{11}^2l_{12} = 3\lambda\beta_1\beta_2\beta_3k_1. \end{cases} \Leftrightarrow \begin{cases} 4\lambda_{22}^3 = -\lambda\beta_1\beta_2\beta_3D^3, \\ 12\lambda_{22}^2\lambda_{12} = -3\lambda\beta_1\beta_2\beta_3k_1D^3. \end{cases}$$

The second identity shows that  $\lambda_{11} = -4\lambda_{22}^2/(\lambda\beta_1\beta_2\beta_3D^2)$ . Hence  $\lambda\lambda_{11}^3\beta_1\beta_2\beta_3 = -4$  from the first identity. We have thus proved from (34) the following result

**Theorem 7.3.** *Let  $C_\phi$  be a smooth plane cubic curve as in Proposition 7.2. Then the discriminant  $\Delta_\phi$  of  $\phi$  satisfies*

$$\Delta_\phi = \frac{2^{16}}{(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})^{12}} \left(\frac{\pi}{\omega_1}\right)^{12} (abc)^8.$$

Let us look at the example when  $\phi$  is given in the Weierstrass form  $-y^2+4x^3-g_2x-g_3$ . In this case,  $\lambda_{11} = \lambda_{22} = 1$  and  $\lambda_{12} = \lambda_{21} = 0$ . We thus recover the classical formula  $\Delta_\phi = 2^{16}(\frac{\pi}{\omega_1})^{12}(abc)^8$ . Since the discriminant of plane cubics is an invariant of weight 12, the factor  $(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})$  in Theorem 7.3 naturally appears as the determinant of the linear transformation (31) which transform a cubic to a Weierstrass form.

From Remark 6.4, we can also treat the other case where the line  $l$  passes through 2-torsion points of  $C_\phi$ . Furthermore, the set of cubics  $\phi$  in the above theorem forms an open dense subset of the space of all ternary cubics and we have thus obtained Theorem 1.6.

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